Addendum: Lyapunov Exponent Calculation

Experiment CP-Ly

Equations of Motion

The system phase $\mathbf{u}$ is written in vector notation as

$$\mathbf{u} = \begin{pmatrix} \theta \\ \omega \\ \phi \end{pmatrix} \tag{1}$$

The equation of motion can be expressed compactly using the vector notation

$$\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u}) \tag{2}$$

where

$$\dot{\mathbf{u}} = \begin{pmatrix} \dot{\theta} \\ \dot{\omega} \\ \dot{\phi} \end{pmatrix} \tag{3}$$

and $\mathbf{G}(\mathbf{u})$ is a 3-vector function of the three dynamic variables.

$$\mathbf{G}(\mathbf{u}) = \begin{pmatrix} G_\theta(\theta,\omega,\phi) \\ G_\omega(\theta,\omega,\phi) \\ G_\phi(\theta,\omega,\phi) \end{pmatrix} \tag{4}$$

For our system

$$\mathbf{G}(\mathbf{u}) = \tag{5}$$

$$\begin{pmatrix} \omega \\ -\Gamma\omega - \Gamma' \mathrm{sgn}\,\omega - \kappa\theta + \mu\sin\theta + \epsilon\cos\phi \\ \Omega \end{pmatrix}$$

Extreme sensitivity to initial conditions manifests itself as an exponential growth in the phase-space separation between two trajectories which start out very near one another. Two points near one another on a Poincaré section are an experimental realization of such a pair. Such nearby points on a Poincaré section typically come at very different times in the trajectory. Nonetheless, a simple (gedanken) time shift allows them to be considered as two separate, but identical, systems simultaneously started and then allowed to evolve. The difference $\delta\mathbf{u}$ between two nearby phase points, $\mathbf{u}$ and $\mathbf{u}'$, is given by

$$\delta\mathbf{u} = \mathbf{u}' - \mathbf{u} \tag{6}$$

The algorithm of Eckmann and Ruelle[1] examines all $\delta\mathbf{u}$ in a small neighborhood around a particular $\mathbf{u}$ on a Poincaré section. The behavior of these $\delta\mathbf{u}$ one time step into the future is then fit to a local model thereby determining the analytic behavior of an *arbitrary* $\delta\mathbf{u}$ over that time step and at that $\mathbf{u}$. This single step analysis is repeated at each point along the trajectory and the results are averaged.

The modelling begins by considering the evolution (time rate of change) of a $\delta\mathbf{u}$ given by

$$\dot{\delta\mathbf{u}} = \dot{\mathbf{u}}' - \dot{\mathbf{u}} \tag{7}$$

which, according to Eq. 2, can be expressed

$$\dot{\delta\mathbf{u}} = \mathbf{G}(\mathbf{u}') - \mathbf{G}(\mathbf{u}) \tag{8}$$

C.Q. 1 *Show that if* $\delta\mathbf{u}$ *is small enough, each component of* $\mathbf{G}(\mathbf{u}')$ *can be approximated by*

[1] J.-P. Eckmann, S. Oliffson Kamphorst, D. Ruelle and S. Ciliberto, "Liapunov exponents from time series." Phys. Rev. A **34**, 4971–4979 (1986).

the first two terms of a Taylor expansion about $\mathbf{u}$ *leading to*

$$\dot{\delta\mathbf{u}} = \mathbf{DG}(\mathbf{u})\delta\mathbf{u} \tag{9}$$

where $\mathbf{DG}(\mathbf{u})$ *is the Jacobian matrix of* $\mathbf{G}$

$$\mathbf{DG}(\mathbf{u}) = \left.\frac{\partial \mathbf{G}}{\partial \mathbf{u}}\right|_{\mathbf{u}} \tag{10}$$

which for our three dimensional phase space would be given by

$$\left.\frac{\partial \mathbf{G}}{\partial \mathbf{u}}\right|_{\mathbf{u}} = \begin{pmatrix} \frac{\partial G_\theta}{\partial \theta} & \frac{\partial G_\theta}{\partial \omega} & \frac{\partial G_\theta}{\partial \phi} \\ \frac{\partial G_\omega}{\partial \theta} & \frac{\partial G_\omega}{\partial \omega} & \frac{\partial G_\omega}{\partial \phi} \\ \frac{\partial G_\phi}{\partial \theta} & \frac{\partial G_\phi}{\partial \omega} & \frac{\partial G_\phi}{\partial \phi} \end{pmatrix} \tag{11}$$

with all derivatives evaluated at $\mathbf{u}$.

For our apparatus, the matrix $\mathbf{DG}(\mathbf{u})$ is given by[2]

$$\mathbf{DG}(\mathbf{u}) = \begin{pmatrix} 0 & 1 & 0 \\ -\kappa + \mu\cos\theta & -\Gamma & -\epsilon\sin\phi \\ 0 & 0 & 0 \end{pmatrix} \tag{12}$$

Equation 9 (with Eq. 10) represents the "local model," a linearization (first order Taylor expansion) of Eq. 2 about one point $\mathbf{u}$. Whereas the full nonlinear description (Eq. 2) is valid everywhere but can not be solved analytically, the local linear model (Eq. 9) is only valid for points in the neighborhood of the one particular point $\mathbf{u}$, but it can be solved analytically. We continue with a brief description of these local, analytic solutions describing the motion of an arbitrary $\delta\mathbf{u}$ in the immediate neighborhood of a particular $\mathbf{u}$.

[2]The derivative of the axle friction term is zero except with respect to ω at $\omega = 0$. It leads to delta function in the $\partial G_\omega/\partial\omega$ term, which is left out for readability.

The matrix elements of $\mathbf{DG}(\mathbf{u})$ are evaluated at the value of $\mathbf{u} = (\theta, \omega, \phi)$ about which the arbitrary $\delta\mathbf{u}$ is centered. Thus, $\mathbf{DG}(\mathbf{u})$ is a constant matrix and Eq. 9 is a set of coupled differential equations with constant coefficients. This kind of system is solved by exponential functions. First, eigenvalues σ_j and eigenvectors $\hat{\mathbf{e}}_j$ are found satisfying

$$\mathbf{DG}\,\hat{\mathbf{e}}_j = \sigma_j \hat{\mathbf{e}}_j \tag{13}$$

The characteristic equation used to find the eigenvalues σ_j is

$$\det|\mathbf{DG} - \sigma\mathbf{I}| = 0 \tag{14}$$

where $\mathbf{I}$ is the identity matrix. For example, with $\mathbf{DG}$ given by Eq. 12, the eigenvalue equation becomes

$$\sigma(\sigma^2 + \Gamma\sigma + \kappa - \mu\cos\theta) = 0 \tag{15}$$

and has the three solutions: $\sigma_0 = 0$ and

$$\sigma_\pm = -\frac{\Gamma}{2} \pm \sqrt{\left(\frac{\Gamma}{2}\right)^2 - \kappa + \mu\cos\theta} \tag{16}$$

As can be seen from this equation, the two eigenvalues $\sigma_\pm$ can both be real and negative, or one can be positive and one negative, or they may occur as complex conjugate pairs with negative real parts. Further note that in all three cases, their sum is given by $-\Gamma$.

The eigenvectors can be found by substituting eigenvalues back into Eq. 13. (Recall they are determined up to an overall multiplier.) In this way, the eigenvector for $\sigma_0 = 0$ is found to be

$$\hat{\mathbf{e}}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{17}$$

and the eigenvectors for the two eigenvalues $\sigma_\pm$ can be taken as

$$\hat{\mathbf{e}}_\pm = \begin{pmatrix} 1 \\ \sigma_\pm \\ 0 \end{pmatrix} \tag{18}$$

There is a unique decomposition of any initial $\delta\mathbf{u}(0)$ into a linear superposition of these eigenvectors

$$\delta\mathbf{u}(0) = \sum \beta_j \hat{\mathbf{e}}_j \tag{19}$$

where the (possibly complex) β_j give the amplitude of the particular eigenvector. Then, the solution to Eq. 9 can be shown to be

$$\delta\mathbf{u}(t) = \sum \beta_j e^{\sigma_j t}\, \hat{\mathbf{e}}_j \tag{20}$$

The particular eigenvalue-eigenvector pair σ_0-$\hat{\mathbf{e}}_0$ gives the expected behavior of a non-zero $\delta\phi$, i.e., it is a constant. The two eigenvectors $\hat{\mathbf{e}}_\pm$ span the θ-ω plane and, like their corresponding eigenvalues, depend on the location of the point $\mathbf{u}$. Where the eigenvalues are real, the behavior of an arbitrary $\delta\mathbf{u}$ is a combination of two exponentials (both decaying, or one decaying and one growing). Where the eigenvalues occur as complex conjugate pairs with negative real parts, the motion of an arbitrary $\delta\mathbf{u}$ is contracting, clockwise rotations (inward spirals) about the trajectory point $\mathbf{u}$.

Time Discretization

The experimental trajectory is available as a set of phase points $\mathbf{u}_n$ at time steps τ apart in time and related to the continuous trajectory $\mathbf{u}(t)$ by $\mathbf{u}_n = \mathbf{u}(n\tau)$.

C.Q. 2 *(a) Show that if the time step τ is small, Eq. 9 implies*

$$\delta\mathbf{u}_{n+1} = \mathbf{DF}_n \delta\mathbf{u}_n \tag{21}$$

where

$$\mathbf{DF}_n = \mathbf{I} + \tau\, \mathbf{DG}_n \tag{22}$$

and $\mathbf{DG}_n = \mathbf{DG}(\mathbf{u}_n)$. (b) Show that

$$\mathbf{DF}_n \hat{\mathbf{e}}_i = e^{\sigma_i \tau} \hat{\mathbf{e}}_i \tag{23}$$

Hint: since τ is small, only the first two terms in the expansion of $e^{\sigma_i\tau}$ need be kept.

Eq. 23 shows that $\mathbf{DF}_n$ has the same eigenvectors as $\mathbf{DG}_n$ with eigenvalues given by $e^{\sigma_i\tau}$.

We will shortly see how to string these $\mathbf{DF}_n$ matrices together to perform a long term evolution of a $\delta\mathbf{u}$. But before doing so, we need a way to determine the $\mathbf{DF}_n$ matrices based on the experimental data.

C.Q. 3 *By always considering points on the same Poincaré section, i.e., $\delta\phi = 0$, the dimensionality of the problem is reduced to two. Show that with $\mathbf{DG}$ given by Eq. 12, a $\delta\mathbf{u}$ with $\delta\phi = 0$ stays a $\delta\mathbf{u}$ with $\delta\phi = 0$, and that the equation for the two remaining components becomes*

$$\begin{pmatrix} \dot{\delta\theta} \\ \dot{\delta\omega} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\kappa + \mu\cos\theta & -\Gamma \end{pmatrix} \begin{pmatrix} \delta\theta \\ \delta\omega \end{pmatrix} \tag{24}$$

Note that Eq. 24 leads to the same two eigenpairs as Eq. 12 ($\sigma_\pm$-$\hat{\mathbf{e}}_\pm$ as given by Eqs. 16 and 18). So we can now consider a two-dimensional version of Eq. 21 to be a discrete-time version of the local linear model, Eq. 9. The following procedure is performed at each time step along a trajectory to determine the local 2×2 $\mathbf{DF}_n$ matrix.

First, a set of points $\mathbf{u}_j$ are found by searching through all points on the Poincaré section containing the trajectory's present phase point $\mathbf{u}_n$ and selecting those points inside an elliptical region satisfying

$$\left(\frac{\delta\theta_j}{\theta_r}\right)^2 + \left(\frac{\delta\omega_j}{\omega_r}\right)^2 < 1 \tag{25}$$

where $\delta\theta_j = \theta_n - \theta_j$, $\delta\omega_j = \omega_n - \omega_j$, and θ_r and ω_r specify the major axes of the ellipse, i.e., the size of the neighborhood around $\mathbf{u}_n$ that will be used in the analysis. Two such sets are at the cursors in the top row of Fig. 1, and shown in a blow-up in the second row.

For the point $\mathbf{u}_n$ and the set of points $\mathbf{u}_j$, a corresponding point $\mathbf{u}_{n+1}$ and set $\mathbf{u}_{j+1}$ are also thereby determined as those points one time step later in their respective trajectories. From these two sets are constructed the two sets of deviations $\delta\mathbf{u}_n = \mathbf{u}_n - \mathbf{u}_j$ and $\delta\mathbf{u}_{n+1} = \mathbf{u}_{n+1} - \mathbf{u}_{j+1}$ one time step forward. Two such sets of deviations after an interval $\tau = 0.1T$ (i.e., 1/10th of a drive period) are shown in the third row. (The bottom row, which shows them after a full period, will be discussed later.) The elongation in one direction and flattening in another is the basic prediction of the local linear model, with the amount of elongation and flattening related to the behavior of the local eigenvalues $\sigma_\pm$ along the path from $\mathbf{u}_n$ to $\mathbf{u}_{n+1}$.

The two sets of deviations $\delta\mathbf{u}_n$ and $\delta\mathbf{u}_{n+1}$ can be used to experimentally determine the matrix $\mathbf{DF}_n$. Expanding Eq. 21 into component equations shows the basic structure needed

$$\begin{aligned} \delta\theta_{n+1} &= DF_{\theta\theta}\,\delta\theta_n + DF_{\theta\theta}\,\delta\omega_n \\ \delta\omega_{n+1} &= DF_{\omega\theta}\,\delta\theta_n + DF_{\omega\omega}\,\delta\omega_n \end{aligned} \tag{26}$$

where DF_{ij} are the matrix elements of $\mathbf{DF}_n$. This suggests that the four unknown matrix elements could be determined as the fitting coefficients of two linear regressions, one each for both $\delta\theta_{n+1}$ and $\delta\omega_{n+1}$ (as the independent or y-variable in each fit) which are regressed onto the two terms $\delta\theta_n$ and $\delta\omega_n$ (a pair of x-variables for each fit).

The actual fitting procedure is slightly different. Letting x represent either θ or ω, the two modified regressions can be expressed

$$\begin{aligned} \delta x_{n+1} &= a_x + DF_{x\theta}\,\delta\theta_n + DF_{x\omega}\,\delta\omega_n \\ &\quad + c_{x\theta\theta}(\delta\theta_n)^2 + c_{x\omega\omega}(\delta\omega_n)^2 \\ &\quad + c_{x\theta\omega}(\delta\theta_n)(\delta\omega_n) \end{aligned} \tag{27}$$

The four coefficients — $DF_{\theta\theta}$, $DF_{\theta\omega}$, $DF_{\omega\theta}$, and $DF_{\omega\omega}$ — are still taken as estimates of the elements of $\mathbf{DF}_n$, but now there are additional terms. The constants a_x take into account random errors in $\mathbf{u}_n$ and $\mathbf{u}_{n+1}$. The quadratic terms (containing c's) model second order terms left out of the analysis. They can be expected to contribute more as the size of the $\delta\mathbf{u}$ region used in the fit increases. Including them (and possibly higher order terms) supposedly improves the accuracy of the DF_{ij} terms.[3]

Note that the ending set $\delta\mathbf{u}_{n+1}$ determined in a prior step is not the starting set for the next step. The deviations often grow and some may not be the smallest possible $\delta\mathbf{u}$ around the next trajectory point. Thus, each new linearization analysis begins with new distance calculations and a new set of $\delta\mathbf{u}$.

As shown next, determining the long term behavior of an arbitrary $\delta\mathbf{u}$ (to get the Lyapunov exponents characteristic of the attractor as a whole) will require stringing together $\mathbf{DF}_n$ matrices along a trajectory lasting thousands of drive periods.

Propagation of an arbitrary $\delta\mathbf{u}$ for one short time period is given by Eq. 21, $\delta\mathbf{u}_{n+1} = \mathbf{DF}_n\delta\mathbf{u}_n$. Propagation for the next time period, from $n+1$ to $n+2$ would be given by $\delta\mathbf{u}_{n+2} = \mathbf{DF}_{n+1}\delta\mathbf{u}_{n+1}$. Combining these two gives a propagation for two time periods $\delta\mathbf{u}_{n+2} = \mathbf{DF}_{n+1}\mathbf{DF}_n\delta\mathbf{u}_n$. Continuing in this way over many time periods N, the propagation becomes

$$\delta\mathbf{u}_{n+N} = \mathbf{DF}_n^N\delta\mathbf{u}_n \tag{28}$$

where

$$\mathbf{DF}_n^N = \prod_{i=0}^{N-1}\mathbf{DF}_{n+i} \tag{29}$$

[3] Henry D.I. Abarbanel, Reggie Brown, John J. Sidorowich, and Lev Sh. Tsimring, "The analysis of observed chaotic data in physical systems," Rev. Mod. Phys. **65**, 1331–1392 (1993).

with higher index matrices multiplying on the left of lower index matrices.

The QR Decomposition

The technique to find the eigenvalues of a long term $\mathbf{DF}^N$ matrix from the individually determined $\mathbf{DF}_n$ matrices is a bit of mathematical wizardry based on something called the QR decomposition. Any matrix $\mathbf{M}$ can be decomposed into a product of two matrices

$$\mathbf{M} = \mathbf{QR} \tag{30}$$

where $\mathbf{Q}$ is an orthonormal matrix (equivalent to a pure rotation for a 2×2 matrix).

$$\mathbf{Q} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \tag{31}$$

and $\mathbf{R}$ is an upper triangular matrix

$$\mathbf{R} = \begin{pmatrix} R^{11} & R^{12} \\ 0 & R^{22} \end{pmatrix} \tag{32}$$

This is the (2×2) QR decomposition. (A computer program determines $\mathbf{Q}$ and $\mathbf{R}$ from the any input matrix $\mathbf{M}$.)

The wizardry continues by writing

$$\mathbf{DF}_n\mathbf{Q}_{n-1} = \mathbf{Q}_n\mathbf{R}_n \tag{33}$$

In other words, before a $\mathbf{DF}_n$ matrix is decomposed, it is first multiplied on the right by the prior decomposition's $\mathbf{Q}$ matrix. (For the first matrix decomposition, the prior $\mathbf{Q}$ is taken as the identity matrix.) Thus the sequence, starting from $\mathbf{DF}_1$ becomes:

$$\begin{aligned} \mathbf{DF}_1 &= \mathbf{Q}_1\mathbf{R}_1 && (34)\\ \mathbf{DF}_2\mathbf{Q}_1 &= \mathbf{Q}_2\mathbf{R}_2 && (35)\\ \mathbf{DF}_3\mathbf{Q}_2 &= \mathbf{Q}_3\mathbf{R}_3 && (36)\\ &\vdots \end{aligned}$$

Now consider the product of the first two $\mathbf{DF}$ matrices; $\mathbf{DF}^2 = \mathbf{DF}_2\mathbf{DF}_1$. Using the first equation above for $\mathbf{DF}_1$ gives:

$$\mathbf{DF}^2 = \mathbf{DF}_2\mathbf{Q}_1\mathbf{R}_1 \tag{37}$$

Using Eq. 35 for $\mathbf{DF}_2\mathbf{Q}_1$ then gives

$$\mathbf{DF}^2 = \mathbf{Q}_2\mathbf{R}_2\mathbf{R}_1 \tag{38}$$

Continuing, the product of the first three matrices becomes:

$$\begin{aligned} \mathbf{DF}^3 &= \mathbf{DF}_3\mathbf{DF}_2\mathbf{DF}_1 \\ &= \mathbf{DF}_3\mathbf{Q}_2\mathbf{R}_2\mathbf{R}_1 \\ &= \mathbf{Q}_3\mathbf{R}_3\mathbf{R}_2\mathbf{R}_1 && (39) \end{aligned}$$

where Eq. 38 was substituted to get to the second line and Eq. 36 to get to the third line.

This technique continues indefinitely so that

$$\mathbf{DF}^N = \mathbf{Q}_N \prod_{n=1}^{N} \mathbf{R}_n \tag{40}$$

where again higher index matrices multiply on the left of lower index matrices.

It is the eigenvalues of $\mathbf{DF}^N$ for N large that are sought. For N large, the effect of the single $\mathbf{Q}$ matrix can be ignored and we can simply find the eigenvalues of the product of the $\mathbf{R}$ matrices. Because the $\mathbf{R}$ matrices are upper diagonal, their eigenvalues are easily shown to be their diagonal elements. Furthermore, the eigenvalues of their product is the product of their eigenvalues. Finally, taking the natural logarithm of the eigenvalues of $\mathbf{DF}^N$ (predicted to be $\lambda_i N\tau$), converts the product to a sum yielding

$$\lambda_i = \lim_{N\to\infty} \frac{1}{N\tau} \sum_{n=1}^{N} \ln R_n^{ii} \tag{41}$$

where the R_n^{ii} are the diagonal elements of the $\mathbf{R}_n$.

Careful consideration must be given to the size of the time step τ. It cannot be too large, but it need not be small. Eq. 21 can be expected to remain valid for larger τ, though it will no longer be given by Eq. 22. It is the Taylor expansion resulting in **DG** that must remain valid throughout the interval, so both the $\delta\mathbf{u}_n$ and $\delta\mathbf{u}_{n+1}$ must remain relatively small compared to the attractor. Because $\delta\mathbf{u}_{n+1}$ grows exponentially with τ, linearly larger values of τ will require exponentially smaller values of $\delta\mathbf{u}_n$. The literature often suggest that the time step be made equal to the drive period. This has the advantage that only phase points from a single Poincaré section are then needed. Figure 1 shows that for our attractors, one full period is sometimes unacceptably long. For the set on the left, the phase points maintain an elliptical shape over an entire drive period. This is not true for the set on the right. (Also note the axes ranges are three times larger in the bottom right figure.) In some regions of the attractor, the phase points distort even more over one drive period. For the group on the left, the fitting model of Eq. 22 would be acceptable for a time interval τ equal to a full period. For the group on the right a smaller interval is needed. As can be seen from the plots in the third row, using $\tau = 1/10$th of a period is acceptable for both sets and is the recommended value for analysis.

This algorithm is implemented in the *Lyapunov* program.

C.Q. 4 *Consider a set of identical chaotic pendulums obeying* $\dot{\mathbf{u}} = \mathbf{G}(\mathbf{u})$ *with* $\mathbf{G}(\mathbf{u})$ *given by Eq. 4. The set has initial conditions filling a small phase space area on a single* θ-ω *plane (of drive phase* ϕ*) as shown in Fig. 2. The area occupied by these systems is given by*

$$A = (\theta' - \theta)(\omega' - \omega) \tag{42}$$

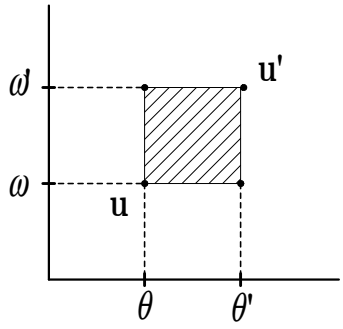

Figure 2: A small area of a θ-ω phase plane for C.Q. 4.

(a) Show that as the systems evolve, the area obeys

$$\frac{d\,A}{dt} = \left(\frac{d\,G_\theta}{d\theta} + \frac{d\,G_\omega}{d\omega}\right) A \tag{43}$$

where the derivatives are evaluated at the point $\mathbf{u} = (\theta, \omega, \phi)$*. Hint: At the lower left corner* $\dot{\theta} = G_\theta(\theta, \omega, \phi)$*. At the lower right corner* $\dot{\theta}' = G_\theta(\theta', \omega, \phi)$*. Use a Taylor expansion of* $G_\theta(\theta', \omega, \phi)$ *about* $\mathbf{u} = (\theta, \omega, \phi)$*. Ditto for the* ω *direction. (b) Show that for our system (with* $\Gamma' = 0$*) the area occupied by the systems decays exponentially everywhere with the same rate* Γ*, i.e.,* $A = A_0 e^{-\Gamma t}$ *(c) This demonstrates that the area occupied by these systems decreases continuously and exponentially everywhere at the same rate. How could the area decrease continuously and leave a Poincaré section with any non-zero extent? How could the area decrease continuously and yet have nearby phase points which diverge exponentially? Consider: If the area decreases everywhere, must all lengths likewise decrease? How could the area decrease while a length increases?*

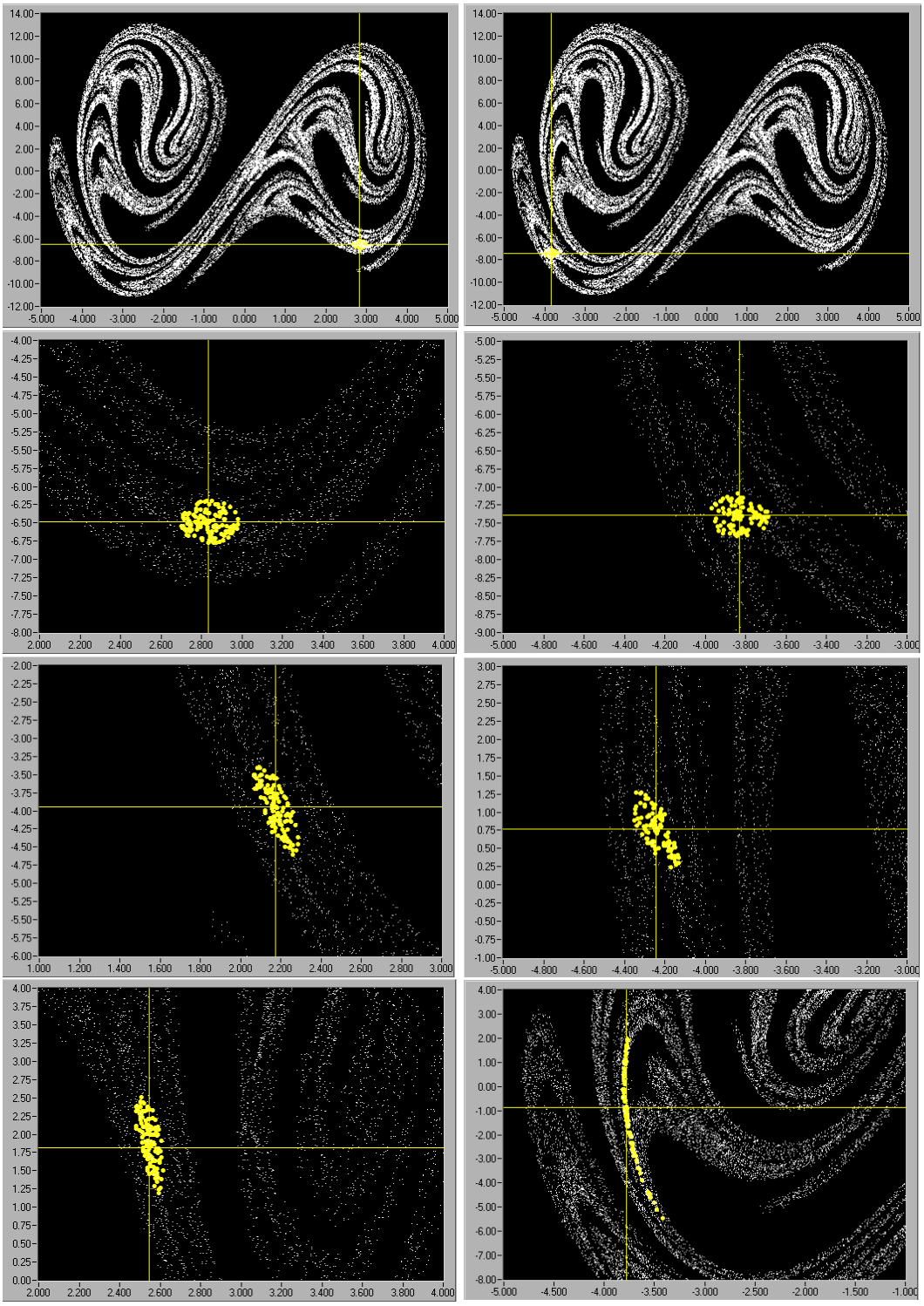

Figure 1: The left and right columns show different behavior for phase points in different areas of the same Poincaré section. The position of the points within the Poincaré section is shown by the cursor in the first row. These regions are blown up in the second row, which shows the two sets (highlighted) as initially selected with radii $\theta_r = 0.15$ rad and $\omega_r = 0.3$ rad/s. The third row shows the points 1/10th of a period later, and the bottom row shows them a full period later. Five of the bottom six plots all have the same size axis ranges — 2 rad for the θ-axis and 4 rad/s for the ω-axis — for the figure on the bottom right they are three times larger in both directions. For all but this bottom right figure, the points are in an elliptical region — the prediction of the local model.