DIMENSIONAL ANALYSIS

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I. SOME EXAMPLES

1. *Strong explosion*

A nuclear bomb goes off at $t = 0$. The energy deposited into the explosion is E_0, the air density before the explosion is ρ_0. Find the dependence of the shock wave front, R, on time t.

Dimensions

$$\begin{aligned} [E_0] &= \frac{ML^2}{T^2}, \\ [\rho_0] &= \frac{M}{L^3}, \\ [R] &= L, \\ [t] &= T \end{aligned}$$

As all three variables of the problem (E_0, ρ_0, and t_0) have independent dimensions, R must be a product of those three

$$R \sim E_0^a \rho_0^b t^c.$$

The dimensions of the left-hand-side and right-hand-side must coincide:

$$L = \left(\frac{ML^2}{T^2}\right)^a \left(\frac{M}{L^3}\right)^b T^c$$

Equating the exponents of like dimensions gives:

$$\begin{aligned} L &: \ 1 = 2a - 3b \\ M &: \ 0 = a + b \\ T &: \ 0 = -2a + c \end{aligned}$$

from which

$$a = 1/5, b = -1/5, c = 2/5$$

Therefore,

$$R \sim \left(\frac{E_0}{\rho_0}\right)^{1/5} t^{2/5}.$$

When this solution is going to work? The size of the bomb, R_0, did not enter (that allowed to form the scaling combination for R), which means that we have effectively treated the bomb as a point source. Every source looks like a point one from the distance much larger than its size. Therefore, our solution is valid for such times that $R \gg R_0$.

2. Diffusion

At $t = 0, N$ particle is born at the origin ($r = 0$). Find the concentration of the particles, $n(r, t)$, at all subsequent times.

Diffusion is described by two macroscopic equations: a) continuity equation, which reflects the particle number conservation

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot \mathbf{j} = 0$$

and Ficks law, which relates the particle current to the concentration gradient

$$\mathbf{j} = -D\vec{\nabla}n,$$

where D is the *diffusion coefficient.* The dimensions of D follow directly from the Ficks law

$$[D] = \frac{L^2}{T}.$$

If our particles move in such a way that the mean free path is ℓ and the mean free time is τ, then a microscopic theory (Boltzmann equation) gives $D = \ell^2/d\tau$, where d is the spatial dimensionality. Substituting Ficks law into the continuity equation results in the diffusion equation

$$\frac{\partial n}{\partial t} = D\nabla^2 n.$$

As time $t = 0$, all the particles were concentrated at $r = 0$, the initial condition for this equation is

$$n(r, 0) = N\delta(r).$$

The total number of particles is conserved at all times. That means the integral of n over the entire space must not depend on time

$$\int d^d r n(r, t) = N.$$

Concentration n is a function of three variables (r, t, and D) :

$$n = f(r, t, D)$$

Out of those three, only two have independent dimension, whereas the dimensions of the third one can be expressed via that of the first two. For example, r and t may be taken as having independent dimensions (L and T, respectively), whereas $[D] = [r]^2/[t] = L^2/T$. It

is more convenient to take the dimensions of D and t as independent ones though. Then, $[r] = [D]^{1/2}[t]^{1/2}$. That means that n can be represented as

$$n = \frac{N}{r^d} F\left(\frac{r}{\sqrt{Dt}}\right).$$

I have chosen r^{-3} (inverse volume in the $d-$dimensional space) to be the overall scale of n. Also, because of number conservation, n must be proportional to N. It is more convenient to re-write the last equation in the following form

$$n = N\frac{(Dt)^{d/2}}{r^d}\frac{1}{(Dt)^{d/2}}F\left(\frac{r}{\sqrt{Dt}}\right) = N\frac{1}{(Dt)^{d/2}}\Phi\left(\frac{r}{\sqrt{Dt}}\right),$$

where the new dimensionless function

$$\Phi(x) \equiv \frac{1}{x^d}F(x).$$

Substituting of the scaling form of n into the diffusion equation helps to reduce this *partial* differential equation into the *ordinary* one. Let's do this for $d = 1$, (diffusion on the line, $r = x$). Denoting

$$\xi \equiv \frac{x}{\sqrt{Dt}},$$

we get

$$\begin{aligned}
\frac{\partial n}{\partial t} &= N\frac{\partial}{\partial t}\frac{1}{(Dt)^{1/2}}\Phi(\xi) = \\
&= -\frac{N}{2D^{1/2}}\frac{1}{t^{3/2}}\Phi(\xi) + N\frac{1}{(Dt)^{1/2}}\frac{d\Phi}{d\xi}\frac{\partial\xi}{\partial t} \\
&= -\frac{N}{2D^{1/2}}\frac{1}{t^{3/2}}\Phi(\xi) - N\frac{1}{(Dt)^{1/2}}\frac{x}{2D^{1/2}t^{3/2}}\frac{d\Phi}{d\xi}. \\
\nabla^2 n &= \frac{\partial^2 n}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}n\right) = \frac{N}{(Dt)^{1/2}}\frac{\partial}{\partial x}\left(\frac{d\Phi}{d\xi}\frac{\partial\xi}{\partial x}\right) \\
&= \frac{N}{(Dt)^{1/2}}\frac{\partial}{\partial x}\left(\frac{d\Phi}{d\xi}\frac{1}{(Dt)^{1/2}}\right) \\
&= \frac{N}{(Dt)^{1/2}}\frac{1}{(Dt)^{1/2}}\frac{d^2\Phi}{d\xi^2}\frac{\partial\xi}{\partial x} \\
&= \frac{N}{(Dt)^{3/2}}\frac{d^2\Phi}{d\xi^2}.
\end{aligned}$$

Diffusion equation reduces to

$$\begin{aligned}
&-\frac{N}{2D^{1/2}}\frac{1}{t^{3/2}}\Phi - N\frac{1}{(Dt)^{1/2}}\frac{x}{2D^{1/2}t^{3/2}}\frac{d\Phi}{d\xi} \\
&= D\frac{N}{(Dt)^{3/2}}\frac{d^2\Phi}{d\xi^2}
\end{aligned}$$

Multiplying both sides of the equation by $(Dt)^{3/2}/ND$ gives

$$-\frac{1}{2}\Phi - \frac{1}{2}\frac{x}{(Dt)^{1/2}}\frac{d\Phi}{d\xi} = \frac{d^2\Phi}{d\xi^2}$$

or

$$\frac{d^2\Phi}{d\xi^2} + \frac{1}{2}\xi\frac{d\Phi}{d\xi} + \frac{1}{2}\Phi(\xi) = 0$$

which can be written as

$$\begin{aligned}\frac{d^2\Phi}{d\xi^2} + \frac{1}{2}\frac{d}{d\xi}(\xi\Phi) &= 0 \rightarrow \\ \frac{d\Phi}{d\xi} + \frac{1}{2}\xi\Phi &= 0 \rightarrow \\ \Phi &= Ce^{-\xi^2/4}.\end{aligned}$$

Thus

$$n(r,t) = \frac{N}{(Dt)^{1/2}}C\exp(-\frac{x^2}{4Dt}).$$

The constant C can be found from, e.g., the conservation law:

$$\begin{aligned}\int_{-\infty}^{\infty} dx n(x,t) &= N \rightarrow \\ NC\int_{-\infty}^{\infty} dx\frac{1}{(Dt)^{1/2}}\exp(-\frac{x^2}{4Dt}) &= N \rightarrow \\ NC\int_{-\infty}^{\infty} d\xi\exp\left(-\xi^2/4\right) &= NC \rightarrow \\ C &= \frac{1}{2\sqrt{\pi}}\end{aligned}$$

The final result is

$$n(r,t) = \frac{N}{2(\pi Dt)^{1/2}}\exp(-\frac{x^2}{4Dt}).$$

A. Scattering cross-section for power-law potentials.

B.

Problem.

Suppose that the potential of a scattering center is

$$V(r) = \frac{A}{r^a}, \tag{1}$$

where r is the distance from the center. Using the dimensional analysis, find an order-of-magnitude estimate for the scattering cross-section.

Solution.

The scattering cross-section, σ, has units of area (L^2) and depends on two parameters: energy of the incident particle, ε, and prefactor A.

$$\begin{aligned} [A] &= [E]L^a \rightarrow \\ L &= \left(\frac{[A]}{[E]}\right)^{\frac{1}{a}} \rightarrow \\ \sigma &\sim \left(\frac{A}{\varepsilon}\right)^{1/a}. \end{aligned}$$

Paradox.

Now the paradox comes. Recall the definition of the scattering cross-section from the Classical Mechanics course

$$d\sigma = 2\pi\rho d\rho, \tag{2}$$

where ρ is the impact parameter. σ is the ratio of the number of particles scattered per unit time by the center to the incoming flux: $\sigma = \dot{N}/J$ ($[\dot{N}] = T^{-1}, [J] = L^{-2}T^{-1} \rightarrow [\sigma] = L^2$). Thus $d\sigma$ is the change in the ration caused by the change in the impact parameter by $d\rho$. Total cross-section

$$\sigma = 2\pi \int_0^? d\rho\rho. \tag{3}$$

I intentionally left the upper limit in the integral undetermined. If the potential had finite range of action. i.e., as in the case of a hard-sphere, the upper limit would be this range, as a particle flying at larger distances from the center would not know about its presence. In our case, however, the potential has infinite range. Therefore, the upper limit is infinity, hence the integral diverges

$$\sigma = 2\pi \int_0^\infty d\rho\rho = \infty. \tag{4}$$

What has gone wrong? Why does an exact calculation gives a physically meaningless result whereas the dimensional analysis produces something finite and meaningful?

Let's get to the root of the divergence. It occurs at large impact parameters when the corresponding scattering angles are small. Let's estimate the dependence of the scattering angle, θ, on ρ for large ρ (small θ). For small θ,

$$\theta \sim \frac{\Delta p_\perp}{p}, \tag{5}$$

where $\Delta p_\perp$ is the change in the transverse momentum of the particle and $p = mv$ is the momentum.

$$\begin{aligned} \Delta p_\perp &\sim F_\perp \Delta t \sim \frac{A}{\rho^{a+1}} \frac{\rho}{v} = \frac{A}{\rho^a v} \rightarrow \\ \theta &\sim \frac{A}{\rho^a m v^2} \rightarrow \rho \sim \left(\frac{A}{\theta m v^2} \right)^{1/a} \end{aligned}$$

(one can see that the units are right). Write σ as

$$\sigma \sim \int d\rho \rho = \int d\theta \left| \frac{d\rho}{d\theta} \right| \rho(\theta) \propto \int d\theta \frac{1}{\theta^{1/a+1}} \frac{1}{\theta^{1/a}} = \int \frac{d\theta}{\theta^{2/a+1}} \tag{6}$$

as the exponent > 1, the integral diverges at the lower limit. Divergence is always the sign of the limited validity of our model. Classical Mechanics is a model of limited validity. A more general description is provided by Quantum Mechanics. CM is OK as far as the QM uncertainties in physical quantities are small. But when the quantities themselves are small uncertainties increase. In particular, CM becomes inapplicable at small scattering angles, that is, precisely where we are having problems with the integral.

How large should be the angle for CM to be still applicable? CM works if the uncertainties both in θ and ρ are small:

$$\delta\theta \ll \theta, \delta\rho \ll \rho. \tag{7}$$

For small angles,

$$\delta\theta \sim \delta p_\perp / p \tag{8}$$

Heisenberg uncertainty relation states that

$$\begin{aligned} \delta p_\perp &\sim \hbar/\delta\rho \gg \hbar/\rho \rightarrow \\ \delta\theta &\gg \hbar/p\rho \end{aligned}$$

On the other hand,

$$\begin{aligned} \theta &\gg \delta\theta \gg \hbar/p\rho \rightarrow \\ \theta &\gg \hbar/p\rho = \hbar/mv\rho = \hbar/L \end{aligned}$$

(angular momentum should be larger than $\hbar$). Now, in CM description

$$\theta \sim \frac{A}{\rho^a m v^2} \tag{9}$$

thus

$$\frac{A}{\rho^a m v^2} \gg \frac{\hbar}{mv\rho} \rightarrow$$
$$\rho^{a-1} \ll \frac{A}{\hbar v}$$

If $a < 1$, CM works at large ρ, if $a > 1$, it works at small ρ, for $a = 1$ (Coulomb potential) the range of applicability does not depend on ρ but only on the parameters ($A = Ze^2 \rightarrow Ze^2/\hbar v \gg 1$ –note that this condition is opposite to that one needs to use Born approximation for the Coulomb potential).

So, CM fails, therefore the cross-section can be determined only by using QM. Thus, the whole body of many centuries work on molecular theory of gases, which was done before QM is wrong. How come that we still use this results?

Divergence in the calculation in an unpleasant thing but it may not necessarily mean a failure of our approach. What if for some physical reason the domain of small angles is irrelevant for our purposes? Then we may not run into the problem with QM. Here is an example. Suppose that instead of measuring the total cross section at a single center,we have plenty of them scattered all over the space, and we do transport experiment: say pump some gas through the medium containing these centers. The relevant quantity for us is then *the distribution function*

$$f(\mathbf{r}, \mathbf{p}, t). \tag{10}$$

It satisfies the *Boltzmann equation (*aka kinetic or transport equation)

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{r}} \cdot \frac{d\mathbf{r}}{dt} + \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{d\mathbf{p}}{dt} = I\{f\} \equiv \text{“collision integral”} \tag{11}$$

Collision integral describes scattering processes leading to changes in particle number with given momentum. There are always two terms in $I = I_{gain} - I_{loss}$. For example, if the probability (per unit time) of scattering from momentum $\mathbf{p}$ into $\mathbf{p}'$ at a stationary scattering center is

$$W_{\mathbf{p}\rightarrow\mathbf{p}'}, \tag{12}$$

then

$$
\begin{aligned}
I_{gain} &= \sum_{\mathbf{p}'} f(\mathbf{r},\mathbf{p}',t) W_{\mathbf{p}'\to\mathbf{p}} \\
I_{loss} &= f(\mathbf{r},\mathbf{p},t) \sum_{\mathbf{p}'} W_{\mathbf{p}\to\mathbf{p}'} \\
&\quad \text{and} \\
I &= \sum_{\mathbf{p}'} f(\mathbf{r},\mathbf{p}',t) W_{\mathbf{p}'\to\mathbf{p}} - f(\mathbf{r},\mathbf{p},t) \sum_{\mathbf{p}'} W_{\mathbf{p}\to\mathbf{p}'}
\end{aligned}
$$

Small angle scattering means that $\mathbf{p} \approx \mathbf{p}'$. Each $W_{\mathbf{p}'\to\mathbf{p}}$ and $W_{\mathbf{p}\to\mathbf{p}'}$ have singularities in this limits. But because gain and loss compensate each other this singularity may be canceled. A more detailed analysis of this equation (given, e.g., in Ashcroft & Mermin, Solid State Physics) shows that the relevant cross-section, which governs the evolution of f, is **not** the total cross-section we tried to determine before, but the *transport cross-section*

$$\sigma_t \equiv \int d\Omega \frac{d\sigma}{d\Omega}(1-\cos\theta) = \int d\theta |\frac{d\rho}{d\theta}|(1-\cos\theta), \tag{13}$$

where $d\Omega$ is the element of the solid angle. For small θ, $1-\cos\theta \approx \theta^2/2$ and the integrand behaves as

$$\frac{\theta^2}{\theta^{2/a+1}} = \frac{1}{\theta^{2/a-1}}. \tag{14}$$

For

$$
\begin{aligned}
2/a - 1 \;&<\; 1 \;\to \\
a \;&<\; 1
\end{aligned}
$$

the integral is convergent and is dominated by $\theta \sim 1$ for which quantum uncertainty is small. Thus for the transport cross-section, CM is fine. It turns out that the total cross-section simply does not enter the classical theory of gases–everywhere you see "cross-section" it is either transport or another one–the cross-section of energy transfer–which also finite for $a < 1$. The case $a = 1$ (Coulomb potential) is a marginal one: σ_t is logarithmically divergent (typical situation in plasma physics). Because ln is a slowly varying function the situation is marginal: small angle-scattering events are important but only in the logarithmic sense. This is part of the reason why the plasma physics is such a complicated subject. There exist an approach in which electron-ion collisions are treated as as large-angle events, the small-angle events being combined into an effective force (friction) transfered to the left-hand-side of the Boltzmann equation.

What we were (unconsciously) estimating via the dimensional analysis was the *transport cross-section*, and for this quantity our result is correct (up to a number, certainly).

II. MAIN THEOREM OF THE DIMENSIONAL ANALYSIS (Π– THEOREM)

Suppose that a physical quantity a is expressed as a function of other physical quantities

$$a = f(a_1, \ldots, a_n) \tag{15}$$

Suppose also that out of n parameters, $a_1, \ldots, a_n$, k parameters, $a_1, \ldots, a_k$ have independent units whereas the units of the rest $n-k$ parameters, $a_{k+1}, \ldots, a_n$ can be expressed via the units of the first k

$$[a_{k+1}] = [a_1]^{p_{k+1}} \ldots [a_k]^{r_{k+1}} \tag{16}$$

$$\ldots \tag{17}$$

$$[a_n] = [a_1]^{p_n} \ldots [a_k]^{r_n} \tag{18}$$

The units of the observed quantity, a, must be expressible via units of the independent parameters, $a_1 \ldots a_k$. Indeed, for expression (15), units of a must be expressible via units of $a_1, \ldots, a_n$. But units of $a_{k+1}, \ldots, a_n$ are expressible via that of $a_1 \ldots a_k$, therefore a must be expressible solely via units of $a_1 \ldots a_k$

$$[a] = [a_1]^p \ldots [a_k]^r \tag{19}$$

Form $n-k+1$ dimensionless combinations

$$\Pi = \frac{a}{{a_1}^p \ldots {a_k}^r}, \tag{20}$$

$$\Pi_1 = \frac{a_{k+1}}{{a_1}^{p_{k+1}} \ldots {a_k}^{r_{k+1}}} \tag{21}$$

$$\Pi_2 = \frac{a_{k+2}}{{a_1}^{p_{k+2}} \ldots {a_k}^{r_{k+2}}} \tag{22}$$

$$\ldots \tag{23}$$

$$\Pi_{n-k} = \frac{a_n}{{a_1}^{p_n} \ldots {a_k}^{r_n}} \tag{24}$$

Expression (15) can be re-written as

$$\Pi = \frac{a}{{a_1}^p \ldots {a_k}^r} = \frac{f(a_1, \ldots, a_n)}{{a_1}^p \ldots {a_k}^r} \tag{25}$$

$$= \frac{f(a_1, \ldots, a_k, \Pi_1 {a_1}^{p_{k+1}} \ldots {a_k}^{r_{k+1}}, \ldots, \Pi_{n-k} {a_1}^{p_n} \ldots {a_k}^{r_n})}{{a_1}^p \ldots {a_k}^r} \tag{26}$$

$$= F(a_1, \ldots, a_k, \Pi_1, \ldots, \Pi_{n-k}). \tag{27}$$

Units of $a_1, \ldots, a_k$ are independent. It means that we can pass to a new unit system in such a way that any of parameters $a_1, \ldots, a_k$, e.g., a_1, is multiplied by an arbitrary factor, whereas the remaining ones are unchanged. The first argument of function F is changed by an arbitrary factor, whereas all the other arguments of F remain unchanged as well as its value Π. Therefore, F does not depend on a_1. Likewise, F does not depend on any of $a_1, \ldots, a_k$. In simple words, *a dimensionless function cannot have arguments which are dimensionfull and have independent units.* Thus

$$f(a_1, \ldots, a_n) = {a_1}^p \ldots {a_k}^r F(\Pi_1, \ldots, \Pi_{n-k}). \tag{28}$$

This statement is known as the *main theorem of dimensionless analysis* (not that I know any other auxiliary theorems).

Example: diffusion from a point source.

$$n = f(r, t, D) \tag{29}$$

In this case, $n = 3$. Out of the three parameters, r, t, D, two, e.g., r, t, have independent units, whereas the units of D are expressible via that of r and t

$$[D] = [r]^2 [t]^{-1} . \tag{30}$$

Thus $k = 2$. It is more convenient to choose t and D as parameters with independent units and express units of r via that of t and D

$$[r] = [D]^{1/2}[t]^{1/2} \tag{31}$$

The units of density can be expressed as

$$[n] = [r]^{-3} = \left[[D]^{1/2}[t]^{1/2}\right]^{-3} = [D]^{-3/2}[t]^{-3/2} \tag{32}$$

Dimensionless parameters

$$\Pi = \frac{n}{(Dt)^{-3/2}} \tag{33}$$

$$\Pi_1 = \frac{r}{\sqrt{Dt}} \tag{34}$$

The main theorem then says that

$$n = (Dt)^{-3/2} F\left(\frac{r}{\sqrt{Dt}}\right). \tag{35}$$

The most important case is when there is only one parameter with units expressible via that of other parameters, i.e., $k = n - 1$. In this case, F is a function of a single variable, which makes analysis particular simple. This is not always possible however. For example, diffusion from a source of finite size is characterized by an additional parameter: the size of the source, r_0. Now

$$n = f(r, r_0, t, D) \tag{36}$$

There are two parameters with independent units. The units of the remaining two are expressible through that of those two. The most general statement one can make is that

$$n = (Dt)^{-3/2} F_1\left(\frac{r}{\sqrt{Dt}}, \frac{r_0}{\sqrt{Dt}}\right). \tag{37}$$

But that would not help us to solve the partial differential equation.

Note that there are no "point sources" or any other idealizations of that kind in Nature. They can only serve as *approximations* valid under certain conditions. In particular, the point-source approximation works if we consider distances much larger that the source size, or equivalently, times much longer than the diffusion time through the source. That means that for long times, the dependence of function F_1 on its second argument fades out and F_1 reduces to F :

$$F_1\left(\frac{r}{\sqrt{Dt}}, \frac{r_0}{\sqrt{Dt}}\right)|_{r_0 \ll \sqrt{Dt}} \approx F\left(\frac{r}{\sqrt{Dt}}\right) \tag{38}$$

Dimensional analysis is most effective when the number of relevant parameters is small (ideally, $n = 3$), and there is only one parameter of units expressible through the other two. Thus it is only good for obtaining *intermediate asymptotics*, that is the forms the solution converge to for values of the argument larger then one characteristic value but smaller that the other one. However, every theoretical result is correct only in the sense of intermediate asymptotics, so the value of dimensional analysis should not be underestimated.

Example: strong explosion. The radius of the shock wave front R at time t after explosion depends on the energy deposition E_0, initial air density ρ_0, and t :

$$R = f(E_0, \rho_0, t). \tag{39}$$

In this case all three parameters have independent units: $n = k = 3$. Therefore function F of the main theorem does not depend on any parameter, that is, it is a constant

$$R = E_0^a \rho_0^b t^c \times \text{const} \tag{40}$$

Equating the units of the left- and right-hand sides of this equation, one gets

$$a = 1/5, b = -1/5, t = 2/5 \tag{41}$$

A. Collapse of experimental data.

Dimensional analysis is also useful for simplifying the analysis of experimental data by collapsing them onto a single curve.

Example: filling of a container.

At the beginning of the century, physical chemists E. Bose, D. Rauert, and M. Bose published a series of experimental investigations based on the following scheme. They measured time τ to fill a container of given volume Q due to the flow of various fluids (water, chloroform, bromoform, mercury, ethyl). The flow of fluids was driven by the pressure drop P at the ends of the pipe. Each fluid had its own density ρ and viscosity coefficient μ. The results were represented as series of curves P vs the inverse filling time (on the semi-log scale). Each curve corresponded to a single fluid. Apart from a rather obvious relation that large pressure leads to shorter filling times, these curves did not reveal much of important information. This publication was noticed by subsequently well-know mathematician von Karman (Born-von Karman boundary conditions in the theory of lattice vibrations). He noticed that pressure can be represented as a function of the other parameters

$$P = f(\tau, Q, \mu, \rho). \tag{42}$$

$$[P] = \frac{F}{L^2} = \frac{Ma}{L^2} = \frac{ML/T^2}{L^2} = \frac{M}{LT^2} \tag{43}$$

$$[\tau] = T \tag{44}$$

$$[Q] = L^3 \tag{45}$$

$$[\rho] = \frac{M}{L^3} \tag{46}$$

Viscosity is a proportionality coefficient between the gradient of the fluid velocity and pressure on the layer parallel to the flow

$$P' = -\mu \frac{\partial v_x}{\partial x} \rightarrow \tag{47}$$

$$[P] = \frac{M}{LT^2} = [\mu]\frac{L}{TL} \rightarrow \tag{48}$$

$$[\mu] = \frac{M}{LT} \tag{49}$$

We see that τ, Q, and μ have independent units whereas units of ρ can be expressed via that of those three:

$$[\rho] = [\mu][\tau][Q]^{-2/3}. \tag{50}$$

Units of pressure can be expressed as

$$[P] = [\mu][\tau]^{-1} \tag{51}$$

Thus, $n = 4$, $k = 3$, and $n - k = 1$.

$$P = \mu\tau^{-1} F\left(\frac{\rho}{\mu\tau Q^{-2/3}}\right) = \mu\tau^{-1} G\left(\frac{\tau}{\rho/\mu Q^{-2/3}}\right) \text{ or} \tag{52}$$

$$\frac{P}{\mu\tau^{-1}} = G\left(\frac{\tau}{\rho/\mu Q^{-2/3}}\right) \tag{53}$$

It means that one plots pressure measured in units of $\mu\tau^{-1}$ vs τ in units of $\rho/\mu Q^{-2/3}$ *all experimental points should collapse onto a single curve described by the dimensionless function* $G(x)$. This function may or may not be known, but even if it is not known, plotting in this way allows one to reduce the multitude of experimental data to a single curve. Once this curve is measured for many fluids and its shape is well-established, one does not have to re-do the measurement for another fluid: simple re-scaling of the parameters will give the answer.

Example.

Determine the temperature distribution in the gas in front of a plane flame assuming that the front is moving at constant speed.

Solution.

The temperature distribution is moving along with the front:

$$T(x,t) = T(x - v_0 t) \tag{54}$$

The equation of heat conduction

$$\frac{\partial T}{\partial t} = \chi \frac{\partial^2 T}{\partial x^2}$$

Substituting (54) into this equation, one gets ($\xi \equiv x - v_0 t$)

$$\begin{aligned} -v_0 \frac{\partial T}{\partial \xi} &= \chi \frac{\partial^2 T}{\partial \xi^2} \rightarrow \\ T &= T_\infty + T_0 \exp\left[-(v_0/\chi)(x - v_0 t)\right], \ x \geq v_0 t. \end{aligned}$$

where T_∞ is the temperature at $+\infty$ (away from the front) and T_0 is the temperature at the front. T_0 is to be found by matching the solutions inside and outside the front and cannot be determined at this level.

Sometimes the exponents of power-laws governing the scaling solutions cannot be found from the dimensional analysis. In those cases they have to be determined as eigenvalues of differential equation which results from substituting of a scaling *Ansatz* into the original equation.

Example: converging shock wave.

One way to produce extremely high magnetic field (known as Bitter's method in the US or Sakharov's method in Russia) is to wrap a metallic cylinder by a layer of explosives. The explosion compresses the cylinder over a very short time. Faraday's law

$$\mathcal{E} = -\frac{d\Phi}{dt} = IR$$

For a well-conducting cylinder, R is small, so in the first approximation $R = 0$. Then $\Phi = const$, or $B(t) \times A(t) = const \rightarrow B$ increases with time. (Those of you who are familiar with superconductivity may notice that in superconductors, where always $R = 0$, this phenomeno is known as flux trapping: the amount of flux trapped by a superconducting cylinder is constant under any deformations of the cylinder. Well, here are just saying that a good metal is not that different from a superconductor). During the explosion a *converging* shock wave is formed.

Question: how can on describe the propagation of the converging shock wave?

At some time t_0 the collapse occurs. One can hope to find the scaling solution for times very close to the collapse time $t \approx t_0$ and $t \leq t_0$, so that the radius of the front is much smaller than the initial radius of the cylinder R_0. Assuming that the radius of the front is of the scaling form

$$R(t) = A(t_0 - t)^a,$$

can one use the dimensional analysis to determine A and a? This problem is similar to that of a diverging shock wave with an important difference: the initial energy deposition, E_0, does not play any role at the late stages of compression. R thus depends only on two parameters (initial air density ρ_0 and time t). There are not enough variables to determine A and a just from dimensional analysis. Still, a scaling solution exists. Making scaling *Ansatz* for relevant hydrodynamic variables: velocity, density, and the ratio of pressure to density (the

square of the sound velocity)

$$v = \frac{r}{t}V(\xi), \rho = \rho_0 R(\xi), c^2 = \frac{p}{\rho} = \frac{r^2}{t^2}Z(\xi),$$
$$\xi \equiv \frac{r}{R(t)},$$

and substituting these expressions into appropriate equations of fluid dynamics, one gets a system of three ordinary differential equations. This system has a solution satisfying a number of physically motivated constraints (continuity of certain parameters through the front, energy conservation, etc) only for *a single value of a*. Thus a is *eigenvalue* of these equations. a can be found only numerically and is, generally speaking, an irrational number. For example, for a monoatomic gas, $a \approx 0.6884\ldots$ .Once the value of a is determined, the dimensions of prefactor A is also known: $[A] = L/T^a$. Note that A is of irrational dimensions! The value of it however cannot be determined from this approach as for the scaling *Ansatz* type, A drops out from the reduced equations for V, R, and Z. Its value can be only found by matching the scaling solution with the non-scaling one, describing the wave propagation at distances $\sim R_0$. This means that the information about the initial conditions is not entirely lost even at very late stages of the propagation. For more details, see *Landau & Lifshits, Fluid mechanics.*

Example: similarity in flows of a viscous fluid. Reynolds number.

Consider a steady flow of some incompressible fluid. This can be, e.g., flow past some object immersed into this fluid. Suppose that the shape of the body is fixed and its described by a single parameter of dimensions of length, L. For a sphere, L is the radius, for an ellipsoid, L can be, e.g., one of the semi-axises (the other semi-axis is then related to L via a dimensionless parameter and thus does not constitute an independent scale). The *Navier-Stokes* equation for an incompressible fluid is

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}\cdot\vec{\nabla}\right)\mathbf{v}\right] = -\vec{\nabla}p + \mu\nabla^2\mathbf{v},$$

where ρ is the density of the fluid, p is the pressure, and μ is the viscosity coefficient. The fluid on its own is characterized by a single parameter $\mu' = \mu/\rho$ ("kinematic viscosity"), which is has the same dimensions (and the same order of magnitude!) as the diffusion coefficient of molecules which the fluid is made of. The unknown quantities are $\mathbf{v}$ and p/ρ (for an incompressible fluid, the density is constant). Let the velocity of the flow far away from the object (where we can control it) is u. Then the flow is characterized by three

parameters

$$u, L, \mu'$$

of dimensions

$$L/T, L, L^2/T,$$

respectively. There is only one dimensionless quantity which can be formed out of these three parameters

$$\mathcal{R} = \frac{uL}{\mu'} = \frac{uL\rho}{\mu},$$

which is called the *Reynolds number.* The velocity distribution in the flow must then be of the form

$$\mathbf{v}(\mathbf{r}) = u\vec{f}\left(\frac{\mathbf{r}}{L}, \mathcal{R}\right),$$

where $\vec{f}$ is some vector function. The dimensions for the pressure is obtained by forming a combination ρu^2. Thus

$$p(\mathbf{r}) = \rho u^2 g\left(\frac{\mathbf{r}}{L}, \mathcal{R}\right).$$

Some integral characteristic of the flow, e.g., the force on the body does not depend on $\mathbf{r}$:

$$F = \rho u^2 L^2 h(\mathcal{R}).$$

Thus, two flow systems (formed by two different fluids passing at different speeds two bodies *of the same shape* but of different sizes) are similar if

a) the Reynolds numbers for two flows are the same,

and

b) spatial distances are rescaled with L.

This observation is the foundation of *modeling* in ship construction: before building an actual-size ship or submarine, engineers first build a model and test it choosing the laboratory conditions in such a way that after rescaling of L and $\mathcal{R}$ they correspond to real-life conditions. Apparently, this did not work too well for Titanic, but who could think of putting an iceberg into equation!

Example: interacting Dirac fermions

Consider a system of 1D fermions. Suppose that without interactions the spectrum is

$$\varepsilon = v_F(|k| - k_F).$$

(This is actually an approximate form of of the normal (parabolic) spectrum $\varepsilon = k^2/2m$ linearized near two Fermi points $\pm k_F$.) In the presence of the electron-electron interactions, the spectrum change to

$$\varepsilon = u\left(|k| - k_F\right).$$

Electrons interact via a pair potential $U(x - x')$. Fourier transform determines the strength of interaction at momentum transfer q :

$$\tilde{U}(q) = \int_{-\infty}^{\infty} dx U(x) e^{iqx}.$$

The simplest model is when only forward scattering is allowed, that is $\tilde{U}(q)$ is finite only for $q = 0$. In this case the problem allows for an exact solution and

$$u = v_F \sqrt{1 + \frac{\tilde{U}(0)}{2\pi\hbar v_F}}.$$

Note that in 1D the dimensions of the Fourier transform is

$$\left[\tilde{U}\right] = L\left[\hbar\right] L^{-1}\left[v\right] = \left[\hbar\right]\left[v\right],$$

therefore the only dimensional combination which can enter the answer is $\tilde{U}/\hbar v_F$. Now we want to solve the problem for the case when backscattering is also allowed: $\tilde{U}(2k_F) \neq 0$. For some reasons, the straightforward solution does not give an answer which satisfies all physical conditions, so we want to see what should be the most general form of u consistent with all requirements. Let's formulate those requirements. First of all, dimensional analysis tells us that

$$u = v_F F\left(\frac{\tilde{U}(0)}{\hbar v_F}, \frac{\tilde{U}(2k_F)}{\hbar v_F}\right),$$

where function $F\left(x, y\right)$ is such that

$$F(x, 0) = \sqrt{1 + x/2\pi}. \tag{55}$$

Another constraint follows from the *Pauli principle* which (for spinless electrons) states

$$\Psi(x)\Psi(x') + \Psi(x')\Psi(x) = 0,$$

which means that

$$\Psi^2(x) = 0.$$

The Hamiltonian of the four fermion interaction

$$H_{int} = \int dx \int dx' \Psi^\dagger(x)\Psi^\dagger(x')U\left(x-x'\right)\Psi(x')\Psi(x).$$

Suppose that $U(x-x') = U_0\delta(x-x')$. Then

$$H_{int} = \int dx \left[\Psi^\dagger(x)\right]^2 \left[\Psi(x)\right]^2 = 0.$$

Thus spinless electrons interacting via the contact potential do not interact at all! The Fourier transform of the contact potential does no depend on q

$$\tilde{U}(q) = \int_{-\infty}^{\infty} dx U_0 \delta(x) e^{iqx} = U_0.$$

Therefore, $\tilde{U}(0) = \tilde{U}(2k_F)$. Effective absence of interactions in this case should mean that $u = v_F$. Hence one more constraint on F

$$F(x,x) = 1.$$

That means that $F(x,y)$ is not really a function of two independent variables, but either

$$F(x,y) = F(x-y), \text{ with } F(0) = 1$$

or

$$F(x,y) = F(\frac{x}{y}), \text{ with } F(1) = 1.$$

The latter option is not appropriate as it means that u is singular either at small x or small y, whereas we should be able to the perturbation theory both for small x (weak forward scattering) and y (weak backscattering). The former one in combination with condition (55) gives

$$F(x,y) = \sqrt{1 + \frac{x-y}{2\pi}},$$

or coming back to u

$$u = v_F\sqrt{1 + \frac{\tilde{U}(0) - \tilde{U}(2k_F)}{2\pi\hbar v_F}}.$$