1. (10 points) The Schrödinger equation for a rigid body that is constrained to rotate about a fixed axis and that has a moment of inertia I about this axis is

$$-\frac{\hbar^2}{2I}\frac{\partial^2\Psi}{\partial\phi^2} = i\hbar\frac{\partial\Psi}{\partial t} \qquad ,$$

where $\Psi(\phi, t)$ is a function of time t and the angle of rotation, ϕ, about the axis. What boundary conditions must be applied to the solutions of this equation? Find the normalized energy eigenfunctions and eigenvalues. Are the eigenvalues real? Is there any degeneracy?

2. (10 points) As you know, Prof. Meisel is "sign" challenged from time to time. In the following list, prove the the equalities with an understanding that the "?" designates a place where the sign may be "+" or "−". You have to resolve the issue in each case and justify your choice.

   (a) $[A, B] \;=\,?\; [B, A]$

   (b) $[A, B]^\dagger \;=\,?\; [B^\dagger, A^\dagger]$

   (c) $[AB, C] \;=\,?\; A[B, C] \;?\; [A, C]B$

   (d) $[A, [B, C]] \;=\,?\; [B, [C, A]] \;?\; [C, [A, B]]$

3. (15 points) Consider a three-dimensional vector space spanned by an orthonormal basis given by $|1\rangle$, $|2\rangle$, $|3\rangle$. Let the vectors $|\alpha\rangle$ and $|\beta\rangle$ be given by

$$|\alpha\rangle = i\,|1\rangle - 2\,|2\rangle - i\,|3\rangle \quad \text{and} \quad |\beta\rangle = i\,|1\rangle + 2\,|3\rangle \quad .$$

   (a) Construct $\langle\alpha|$ and $\langle\beta|$ in terms of the given basis vectors.

   (b) Find $\langle\alpha|\beta\rangle$ and $\langle\beta|\alpha\rangle$. Confirm or deny that $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$.

   (c) Find all nine matrix elements of the operator $\hat{A} \equiv |\alpha\rangle\langle\beta|$, in the given basis, and construct the 3×3 matrix $\mathbf{A}$. Is it hermitian?

4. (15 points) The Hamiltonian of a three state system is given by

$$\mathbf{H} = \hbar\omega \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and has eigenstates

$$|\psi_a\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{2\pi i/3} \\ 1 \\ e^{-2\pi i/3} \end{pmatrix} , \quad |\psi_b\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \text{ and } |\psi_c\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{-2\pi i/3} \\ 1 \\ e^{2\pi i/3} \end{pmatrix} .$$

(a) What are the corresponding eigenvalues E_a, E_b, and E_c?

(b) If the system is initially in the $|\psi_i\rangle$ state and an energy measurement is made, what are the possible outcomes of the measurements and the probabilities of those outcomes? Take

$$|\psi_i\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} .$$

(c) Instead of making an energy measurement, let the system evolve with time. What is $|\psi(t)\rangle$?

(d) At a time t, what is the probability of being in the state $|\psi_i\rangle$?

(e) Now consider an energy measurement at a time t. What are the possible outcomes of the measurements and the probabilities of those outcomes?