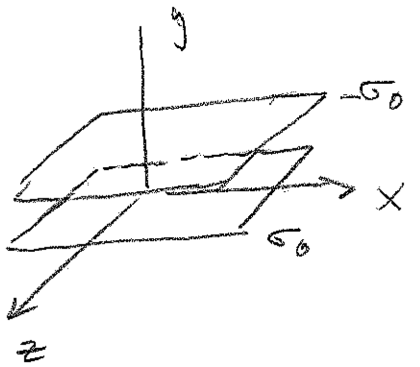
1. (Transforming EM fields.)

A parallel-plate capacitor with one plate in the xz plane with areal charge density σ_0 in its rest frame S, the other located at $y = d$ with areal charge density $-\sigma_0$ is moving to the right along the x axis at speed v as viewed by you in frame $\bar{S}$.

(a) (2 pts.) What electric field do you measure (give all vector components)?



In rest frame $\qquad \gamma = \dfrac{1}{\sqrt{1-v^2/c^2}}$

$$\vec{E} = \vec{E}_\perp = \frac{\sigma_0}{\varepsilon_0}\hat{y}$$

Lorentz transform $\quad \bar{E}_\perp = \gamma E_\perp$

$$\Rightarrow \quad \bar{E}_y = \gamma E_y = \frac{\gamma\sigma_0}{\varepsilon_0}$$

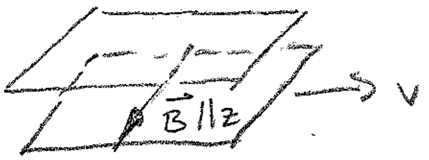
$$\bar{E}_z = \gamma E_z = 0$$

$$\bar{E}_x = \bar{E}_\parallel = E_\parallel = 0$$

(b) (2 pts.) What magnetic field do you measure (give all vector components)

Sheet currents $\quad$ top / bottom plate $\quad \vec{K}_\pm = \mp\sigma v\hat{x}$ in $\bar{S}$!

$B = 0$ above $\qquad = \mp\gamma\sigma_0 v\hat{x}$



$$\Rightarrow \quad \bar{B}_z = \mu_0\sigma v = \mu_0\gamma\sigma_0 v$$

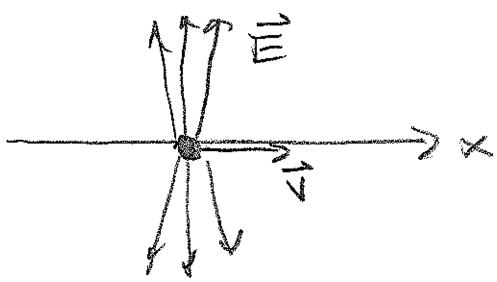
$$\bar{B}_x = \bar{B}_y = 0$$

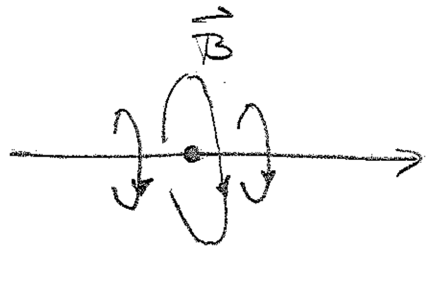
$B = 0$ below

[Recall $\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{encl} \Rightarrow \vec{B} = \mu_0 K/2$ per sheet. Contributions from 2 sheets cancel outside capacitor.]

Now forget the plates and consider a point charge moving at constant velocity v along the x axis as viewed from the frame $\bar{S}$.

(c) (2 pts.) Sketch the electric and magnetic field lines of the moving charge assuming v is close to c.





Radial but bunched $\perp$ to direction of motion

Azimuthal

$$\vec{B} = \frac{1}{c^2}(\vec{v}\times\vec{E})$$

(d) (2 pts.) Find the electric field $\bar{\mathbf{E}}$ in $\bar{S}$ at time $\bar{t} = 0$ by Lorentz transforming from the fields in the charge's rest frame S. Be sure to express $\bar{\mathbf{E}}$ in terms of the $\bar{S}$ coordinates $\bar{x}, \bar{y}, \bar{z}$.

Rest frame $\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r} = \frac{1}{4\pi\varepsilon_0 r^3}(x, y, z)$; $\vec{B} = 0$

$\bar{E}_x = E_x = \frac{1}{4\pi\varepsilon_0} \frac{x}{r^3}$; $\bar{E}_y = \gamma(E_y - vB_z) = \frac{\gamma y}{4\pi\varepsilon_0 r^3}$

$\bar{E}_z = \gamma(E_z + vB_y) = \frac{\gamma z}{4\pi\varepsilon_0 r^3}$

coordinates in $\bar{S}$: $x = \gamma(\bar{x} + v\underbrace{\bar{t}}_{0})$ $y = \bar{y}$ · $z = \bar{z}$

$$\Rightarrow \quad \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{\gamma(\bar{x}, \bar{y}, \bar{z})}{[(\gamma\bar{x})^2 + \bar{y}^2 + \bar{z}^2]^{3/2}}$$

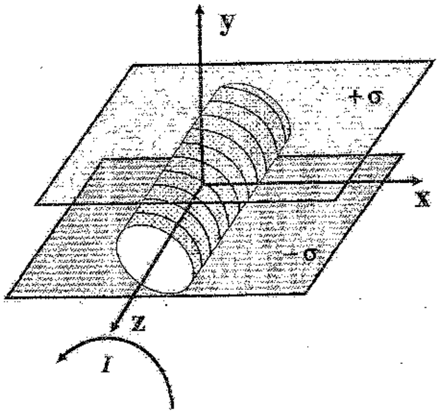
(e) (2 pts.) Find the magnetic field $\bar{\mathbf{B}}(\bar{x}, \bar{y}, \bar{z})$ at $\bar{t} = 0$ in the frame $\bar{S}$.

Similarly $\bar{B}_x = B_x = 0$ $\bar{B}_y = \gamma(B_y + \frac{v}{c^2} E_z) = \frac{v\gamma\bar{z}}{c^2 4\pi\varepsilon_0 [\ ]}$

$$\Rightarrow \bar{B}_y = \frac{v\gamma\bar{z}}{c^2 4\pi\varepsilon_0 [(\gamma\bar{x})^2 + \bar{y}^2 + \bar{z}^2]^{3/2}} = \frac{\mu_0 v\gamma\bar{z}}{4\pi[\ ]^{3/2}}$$

$$\bar{B}_z = \gamma(B_z - \frac{v}{c^2} E_y) = \frac{-\mu_0 v\gamma\bar{y}}{4\pi[\ ]^{3/2}}$$

2. Between the sheets. An infinite solenoid with radius R, n turns per unit length and carrying current I (going counterclockwise in figure as shown) is centered on the z axis. It is sandwiched between two infinite sheets of charge separated by a distance d with surface charge density σ for the $y = +d/2$ plane and $-\sigma$ for the $y = -d/2$ plane as shown.



(a) (2 pts.) Compute **E** and **B** throughout *all* space.

$$\vec{E} = \begin{cases} \frac{-\sigma}{\varepsilon_0}\hat{y} & \text{between planes } \frac{-d}{2} \le y \le \frac{d}{2} \\ 0 & \text{elsewhere} \end{cases}$$

$$\vec{B} = \begin{cases} \mu_0 n I \hat{z} & s^2 = x^2 + y^2 \le R^2 \\ 0 & \text{elsewhere} \end{cases}$$

$$T_{ij} = \varepsilon_0\left(E_iE_j - \frac{1}{2}E^2\delta_{ij}\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$$

(b) (3 pts.)Compute the Maxwell stress tensor in *all* regions of space.

N.B. (A) Inside solenoid $E, B \neq 0$; (B) outside solenoid but between plates $E \neq 0$, $B = 0$; (C) outside plates and solenoid $E = B = 0$.

(A) $T_{xx} = -\frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = -u$; $T_{yy} = \frac{1}{2}\varepsilon_0 E^2 - \frac{1}{2\mu_0}B^2$

$T_{zz} = -\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2$; $T_{ij} = 0$ if $i \neq j$

(B) $T_{xx} = -\frac{1}{2}\varepsilon_0 E^2$; $T_{yy} = \frac{1}{2}\varepsilon_0 E^2$; $T_{zz} = -\frac{1}{2}\varepsilon_0 E^2$; $T_{ij} = 0$ if $i \neq j$

(C) $T_{ij} = 0 \quad \forall\, i,j$

(c) (2.5 pts.)Where in space is momentum stored in the electromagnetic fields? Give the value of the momentum density $\wp(x, y, z)$ in that region.

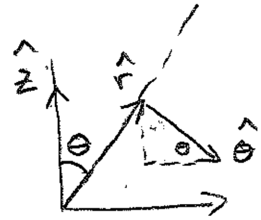
Momentum density $\vec{p}_{em} = \mu_0 \varepsilon_0 \vec{S}$

$= \varepsilon_0 (\vec{E} \times \vec{B}) = \varepsilon_0\left(\frac{-\sigma}{\varepsilon_0}\mu_0 n I\,\hat{x}\right)$ inside solenoid only

$= \mu_0 I \sigma n\,\hat{x}$

3. (Dipole radiation)

(a) (3 pts.) The vector potential of an oscillating electric dipole is



$$\mathbf{A}(r, \theta, t) = -\frac{\omega\mu_0 p_0}{4\pi r}\sin[\omega(t - r/c)]\,\hat{z}$$

where p_0 is the dipole moment amplitude and ω is the angular frequency of oscillation. Calculate $\nabla \cdot \mathbf{A}$ to lowest order for $r \gg \lambda \gg d$, where d is the size of the dipole. Now use the Lorentz gauge to calculate $V(r, \theta, t)$.

Spherical coords. $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$

$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\partial_r(r^2 A_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\partial_\phi A_\phi$

$= -\frac{\omega\mu_0 p_0}{4\pi}\left(\frac{\cos\theta}{r^2}\partial_r\left(r\sin[\omega(t - r/c)]\right) - \frac{1}{r^2}\sin[\omega(t - r/c)]\,\partial_\theta \sin^2\theta\right)$

$= -\frac{\omega\mu_0 p_0}{4\pi}\left(\frac{\cos\theta}{r^2}\sin[\;] - \frac{\omega\cos\theta}{rc}\cos[\;] - \frac{1}{r^2}\sin[\;]\,2\sin\theta\cos\theta\right)$

Neglect 1st + 3rd terms — higher order in $\frac{1}{r}$

So $\vec{\nabla}\cdot\vec{A} = \frac{\mu_0 p_0 \omega^2}{4\pi r c}\cos\theta\cos[\omega(t - r/c)]$

Lorentz gauge: $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$

So $V = -c^2 \int^t \vec{\nabla}\cdot\vec{A}(t')dt'$

$= -\frac{\mu_0 p_0}{4\pi r} c\omega^2 \cos\theta \int^t dt' \cos[\omega(t' - r/c)]\, dt'$

$= -\frac{\mu_0 p_0}{4\pi r} c\omega\cos\theta \sin[\omega(t - r/c)] + \text{const.}$

$= \frac{-p_0}{4\pi\epsilon_0}\frac{\omega}{c}\frac{\cos\theta}{r}\sin[\omega(t - r/c)]$

(b) (2 pts.) An insulating circular ring (radius b) lies in the $x-y$ plane, centered at the origin. It carries a linear charge density $\lambda = \lambda_0 \sin\phi$, where λ_0 is a constant and ϕ is the azimuthal angle around the ring. It is now set spinning at angular velocity ω. What is the dipole moment $\mathbf{p}$ as a function of time?

at $t=0$ dipole moment $\equiv \vec{p} \equiv \int \vec{r}'\rho(\vec{r}')\,d\tau'$

charge density on ring $\rho(\vec{r}')d\tau' = \lambda b\, d\phi'$
$= \lambda_0 b \sin\phi'\, d\phi'$

$\vec{r}' = b(\cos\phi'\hat{x} + \sin\phi'\hat{y})$

$\boxed{\frac{1}{2\pi}\int_0^{2\pi}\sin^2\phi\, d\phi = \frac{1}{2}}$

$\vec{p} = \lambda_0 b^2 \int_0^{2\pi} d\phi' (\underbrace{\cos\phi'\sin\phi'}_{0}\hat{x} + \sin^2\phi'\hat{y})$

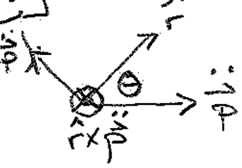
$= \lambda_0 \pi b^2 \hat{y}$ so $\vec{p}(t) = \pi\lambda_0 b^2(\cos\omega t\,\hat{y} - \sin\omega t\,\hat{x})$ (counter clockwise)

(c) (2.5 pts.) The leading radiation fields for an arbitrary source are

$$\mathbf{E}(r,\theta,t) = \frac{\mu_0}{4\pi r}[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\ddot{\mathbf{p}})],$$
$$\mathbf{B}(r,\theta,t) = -\frac{\mu_0}{4\pi r c}[\hat{\mathbf{r}}\times\ddot{\mathbf{p}}],$$

where $\ddot{\mathbf{p}}$ is $d^2\mathbf{p}/dt^2$. Find the total power radiated.

$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{\mu_0}{16\pi^2 c r^2}\underbrace{[(\hat{r}\times\ddot{\vec{p}})\times(\hat{r}\times(\hat{r}\times\ddot{\vec{p}}))]}_{\propto\, \hat{r}\, ?}$



$= \frac{\mu_0 \ddot{p}^2}{16\pi^2 c}\frac{\sin^2\theta}{r^2}\hat{r}$

$P = \int \vec{S}\cdot d\vec{a} = \frac{\mu_0 \ddot{\vec{p}}^2}{16\pi^2 c}\underbrace{\int_0^\pi d\theta\, \sin^3\theta}_{8\pi/3} = \frac{\mu_0 \ddot{p}^2}{6\pi c} = \boxed{\frac{\mu_0 \pi \lambda_0^2 b^4 \omega^4}{6c}}$

4. (Optics)

(a) (3 pts.) A laser pulse has a duration of Δt=10^{-14} s, a total energy of 10^{-2}J, and is focussed to a circular spot 10^{-6} m in radius. Find the intensity in W/m^2 and average electric field in V/m of the laser pulse. If the laser pulse strikes a perfectly absorbing surface, what is the average radiation pressure in N/m^2 during the pulse? (You may assume the pulse is a square wave–fields are zero, then nonzero but constant for a time Δt, then zero again.)

$$I = \frac{\langle P \rangle}{A} = \frac{10^{-2} J/(10^{-14} s)}{\pi \times 10^{-12} m^2} = \frac{1}{\pi} 10^{24} \frac{W}{m^2} = 3.2 \times 10^{23} \frac{W}{m^2}$$

$$\| \quad \frac{1}{2} c \varepsilon_0 E_0^2 \Rightarrow E_0 = \sqrt{\frac{2I}{\varepsilon_0 c}} = \sqrt{\frac{2(3.2\times 10^{23})}{(8.85\times 10^{-12})(3\times 10^{8})}}$$

$$= 1.6 \times 10^{13} \text{ V/m}$$

$$P_{rad} = I/c = 1.1 \times 10^{15} \text{ N/m}$$

(b) (2pts.)The complex wave number of a simple atomic system consisting of electrons bound to a single type of atomic nucleus by a spring with spring constant k_{spring}, fundamental frequency $\omega_0 = \sqrt{k_{spring}/m}$, and damping constant γ is

$$k(\omega) = \frac{\omega}{c}\sqrt{\epsilon_r} \simeq \frac{\omega}{c}\left[1 + \frac{Ne^2}{2m\epsilon_0}\left(\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}\right)\right].$$

How are the phase and group velocities v_p and v_g related to $k(w)$?

$$V_g = \frac{\partial \omega}{\partial k} \qquad V_p = \frac{\omega}{k'} \qquad \text{where } k' = \text{Re}\, k$$

(c) (2.5 pts.) Assuming negligible damping, compute the phase and group velocities, and show that $v_g < c$ always, and that v_p may be greater than or less than c.

"Negligible damping" $\Rightarrow \gamma \ll \omega, \omega_0$

$$\Rightarrow k(\omega) \simeq \frac{\omega}{c}\left[1 + \frac{Ne^2}{2m\varepsilon_0}\frac{1}{\omega_0^2 - \omega^2}\right] \equiv k' \quad \text{purely real}$$

$$\Rightarrow V_p = \frac{\omega}{k'} = \frac{c}{1 + \frac{Ne^2}{2m\varepsilon_0}\frac{1}{\omega_0^2-\omega^2}}$$

2nd term in denominator can be < 0 or > 0

$\Rightarrow V_p > c$ or $V_p < c$

$$\frac{\partial k}{\partial \omega} = \frac{1}{c}\left(1+\frac{Ne^2}{2m\varepsilon_0(\omega_0^2-\omega^2)}\right)+\frac{2\omega^2}{c}\frac{Ne^2}{2m\varepsilon_0}\frac{1}{(\omega_0^2-\omega^2)^2}$$

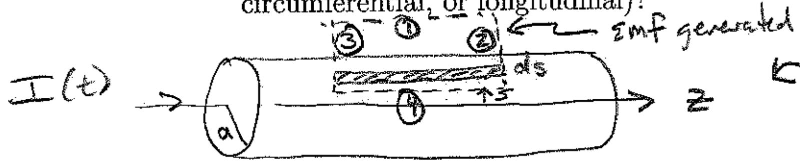
$$= \frac{1}{c} + \frac{1}{c}\frac{Ne^2}{2m\varepsilon_0}\frac{1}{(\omega_0^2-\omega^2)^2}\left[\omega_0^2-\omega^2+2\omega^2\right]$$

$$= \frac{1}{c}\left(1+\frac{Ne^2}{2m\varepsilon_0}\frac{\omega_0^2+\omega^2}{(\omega_0^2-\omega^2)^2}\right)$$

$$\Rightarrow V_g = \frac{c}{1+\frac{Ne^2}{2m\varepsilon_0}\left(\frac{\omega_0^2+\omega^2}{(\omega_0^2-\omega^2)^2}\right)} < c$$

5. (Coax) An alternating current $I = I_0\cos(\omega t)$ flows down a long straight wire and returns along a coaxial conducting tube of radius a. Both the wire and the tube have negligible thickness.

(a) (2 pts.) In what direction does the induced electric field point (radial, circumferential, or longitudinal)?



$\vec{B}$ = "circumferential" $\propto \hat{\theta}$ between wire + tube

Time-changing $\vec{B}$ generates EMF around loop shown. (Faraday) $\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt}$. Field can't be radial or circumferential or EMF = 0 $\Rightarrow$ it would cancel on 2 legs of loop, can't be "longitudinal" i.e. $\vec{E}_{ind} \propto \hat{z}$

(b) (2 pts.) Assuming that the field goes to zero as $s \to \infty$., find $\mathbf{E}(s,t)$, ignoring retardation effects.

Flux $\Phi = \int_s^a ds' \frac{\mu_0 I}{2\pi s'}\cdot \underset{\text{length}}{l} = \frac{\mu_0 I}{2\pi} l \ln\frac{a}{s}$

EMF $\mathcal{E} = \underset{①②③④}{\oint \vec{E}\cdot d\vec{l}} = 0 + 0 + 0 + E\cdot l = -\frac{d}{dt}\frac{\mu_0 I}{2\pi} l \ln\frac{a}{s}$

$$\boxed{\vec{E} = \frac{\mu_0}{2\pi}\underbrace{\omega I_0 \sin\omega t}_{-\frac{dI}{dt}} \ln\frac{a}{s}\hat{z}}$$

(c) (2 pts.) Find the displacement current $\mathbf{J}_d$ everywhere in space.

$$\vec{J_d} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{\varepsilon_0 \mu_0 I_0 \omega^2}{2\pi} \cos\omega t \ln\frac{a}{s}\,\hat{z}$$

only inside coax

(d) (1.5 pts.) Integrate $\mathbf{J}_d$ to get I_d. You may need $\int x \ln x\, dx = x^2(2\ln x - 1)/4$. What is the ratio I_d/I?

$$I_d = \int \vec{J_d}\cdot d\vec{a} = \frac{\varepsilon_0 \mu_0 I_0 \omega^2}{2\pi} \cos\omega t \int_0^a (2\pi s\, ds) \ln\frac{a}{s}$$

$$= \varepsilon_0 \mu_0 I_0 \omega^2 \cos\omega t \underbrace{\int_0^a (s\ln a - s\ln s)}_{\frac{a^2}{2}\ln a - \left(\frac{s^2}{4}(2\ln s - 1)\right)\Big|_0^a}$$

$$= \boxed{\frac{a^2}{4} \varepsilon_0 \mu_0 I_0 \omega^2 \cos\omega t}$$

(e) (Extra credit 2 pts) Take $a = 2\text{mm}$. How high does the frequency have to be to produce $I_d/I = 10^{-2}$?

$I = I_0 \cos\omega t$ so

$$\frac{I_d}{I} = \frac{\varepsilon_0 \mu_0 \omega^2 a^2}{4} = \left(\frac{\omega a}{2c}\right)^2 = 0.01$$

$$\frac{\omega}{2\pi} = \frac{2c}{2\pi a}\cdot(0.1) = \frac{\cancel{2}(3\times 10^8)}{\cancel{2}\pi(2\times 10^{-3})}$$

$$\cong 0.5\times 10^{11}\ \text{Hz}$$

$$= 5\times 10^{10}\ \text{Hz} = 50\ \text{GHz}$$

6) Drop tildes "~" and take Re parts at end:

Note $\vec{E}_{1R}$ is full reflected wave in I on I

a) I : $\vec{E}_I = \vec{E}_{1I}\, e^{i(kz-\omega t)} + \vec{E}_{1R}\, e^{i(-kz-\omega t)}$ ← $k=\omega/c$ (air) →

II $\vec{E}_{II} = \vec{E}_{2T}\, e^{i(k_2 z-\omega t)} + \vec{E}_{2R}\, e^{-i(k_2 z+\omega t)}$

III $\vec{E}_{III} = \vec{E}_{3T}\, e^{i(kz-\omega t)}$

Note $k_1 = k_3 = k = \omega/c$ since we're told $n_{1,3}=1$,
$k_2 = k/n$

b) Gen'l B.C. are $D^\perp, B^\perp$ continuous
$E^{\parallel}, H^{\parallel}$ continuous

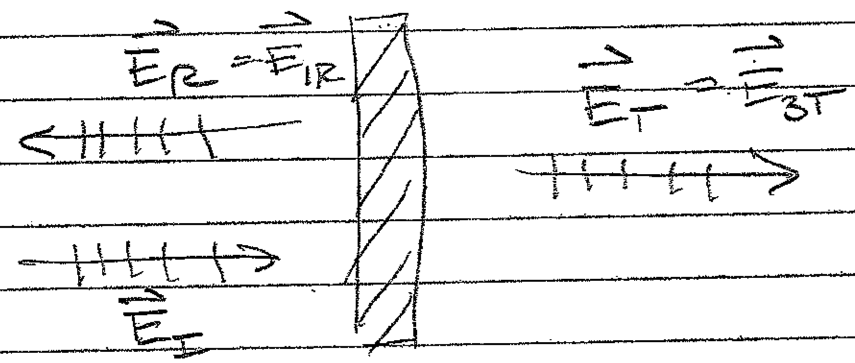
Here due to normal incidence we have no $\perp$ components; also $H^{\parallel} = B^{\parallel}$ since $\mu = \mu_0$.

c) at I-II interface $z=0$

c1) $\vec{E}_{1I} + \vec{E}_{1R} = \vec{E}_{2T} + \vec{E}_{2R}$

at $z=d$ c2) $e^{ik_2 d}\,\vec{E}_{2T} + e^{-ik_2 d}\,\vec{E}_{2R} = e^{ikd}\,\vec{E}_{3T}$

Note there are 2 interfaces but the total reflection coeff. R is the reflected wave in region I (which includes the reflected wave from the interface at d), similarly the transmission makes the final transmission amplitude



So we would solve for

$$R = \left|\frac{E_{1R}}{E_{1I}}\right|^2 \qquad T = \left|\frac{E_{3T}}{E_{1I}}\right|^2$$

To solve we have 4 eqns with 5 unknowns: $E_{1I}, E_{1R}, E_{2T}, E_{2R}, E_{3T}$

(eqns are c1) and c2) with magnetic analogs with $B_\alpha = E_\alpha / v_\alpha$)

So we can solve for all 4 reflected + transmitted amplitudes in terms of E_{1I}