PHY4324–Electromagnetism II

Fall 2011

Final exam – 2 hours

Dec. 13, 2011

No other materials except calculators allowed. If you can't do one part of a problem, solve subsequent parts in terms of unknown answer–define clearly. **Do 4 of 6 problems, CLEARLY indicating which you want graded by circling the problem number!**. *Each problem is worth 10 pts., for a maximum of 40 points, which will be normalized to 30 on final score.*

The following expressions may be useful:
$\epsilon_0 = 8.85 \times 10^{-12}\mathrm{C}^2/\mathrm{N\text{-}m}^2$; $\mu_0 = 4\pi \times 10^{-7}$ N/A^2

For a conductor:

$$\tilde{k} = k + i\kappa \quad ; \quad k = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2} \quad ; \quad \kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}$$

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1. **Electromagnetic waves in conductors.**
   (a) (3 pts.) A given material has a DC conductivity of $\sigma = 5 \times 10^5$ $(\Omega\text{-m})^{-1}$, permittivity $\epsilon = 10\epsilon_0$, and permeability $\mu = \mu_0$. For $\omega = 5 \times 10^{15}$ Hz, compute the skin depth d
   (b) (3 pts.) For the same material, find the index of refraction n.
   (c) (2 pts.) Compute the *numerical* value of the reflection coefficient R assuming the light is normally incident on the surface.
   (d) (2 pts.) An EM wave of angular frequency ω traveling in a vacuum is normally incident on the surface of a material of conductivity σ, permittivity ϵ and permeability $\mu = \mu_0$. Find an analytical expression for the compex wavenumber $\tilde{k}$ in the limit $\sigma \ll \omega\epsilon$. From this, derive an analytical expression for the reflection coefficient R in this limit.

2. **Gauge transformations**:
   (a) (3 pts.) The scalar and vector potentials for a charge at rest are given by $V = q/(4\pi\epsilon_0 r)$ and $\mathbf{A} = 0$. To these potentials satisfy Coulomb gauge? Lorentz gauge?
   (b) (3 pts.) Transform the potentials in part (a) using the gauge function $\lambda(\mathbf{r}, t) = qt/(4\pi\epsilon_0 r)$. What are the new potentials V' and $\mathbf{A}'$.
   (c) (2 pts.)Do the new potentials satisfy Coulomb gauge? Lorentz gauge?
   (d) (2 pts.) Show that it is always possible to find a gauge where $V = 0$, but not necessarily where $\mathbf{A} = 0$.

3. **Radiation from an electron bound to an infinitely heavy nucleus.** Consider an electron, charge $-e$, mass m, bound to a nucleus by a "spring" with spring constant k. Assume that the electron senses a complex electric field of the form $\tilde{E}(t) = E_0 e^{i\omega t}$, and that a damping force $F_d = -m\gamma\frac{dx}{dt}$ acts on the electron/spring system.
   (a) (3 pts.) Write down the equation of motion of the electron
   (b) (3 pts.) Assume a complex position $\tilde{x} = \tilde{x}_0 e^{i\omega t}$. Solve for $\tilde{x}(t)$ and the complex acceleration $\tilde{a}(t)$.
   (c) (2 pts.) Define the time-dependent dipole moment of this system.
   (d) (2 pts.) Write an expression for the total, time averaged electric dipole radiation power. Be sure to take the real part of the dipole moment operator first.

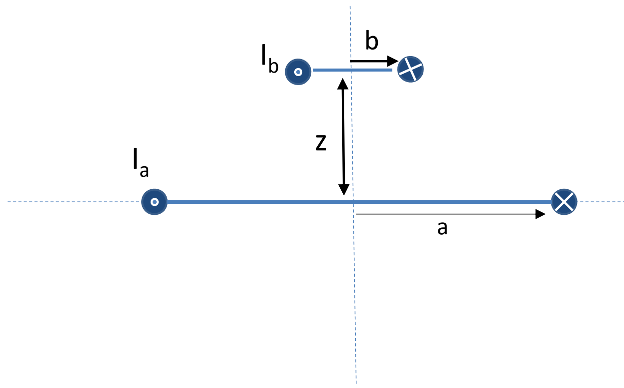


FIG. 1: Side view of 2 loops

4. A small circular loop (radius b) lies a distance z above the center of a 2nd circular loop (radius a). You may assume $a \gg b$ as indicated in the figure.

   (a) (4 pts.) Find the magnetic field a distance z above the center of the circular loop on the z axis *only*.

   (b) (4 pts.) Find the mutual inductance M between the two loops.

   (c) (2 pts.) Now the current in loop a varies as $I_a = I_0 \cos \omega t$. Find the induced E-field at loop b.

5. (Line charge) Consider an infinitely long, straight, infinitesimally thin line charge which has charge density per unit length λ in its own rest frame.

   (a) (3pts.) Write down the **E** and **B** fields in this frame, and construct the electromagnetic field tensor $F^{\mu\nu}$.

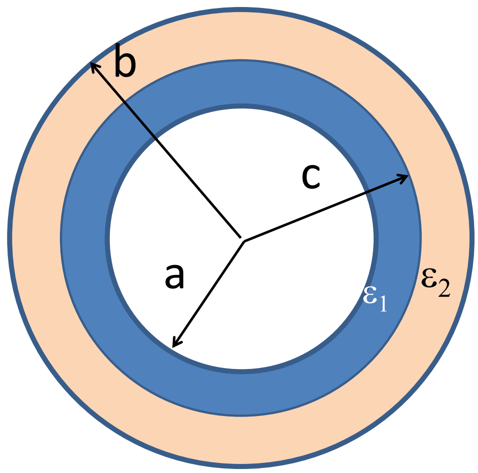
   (b) (3 pts.) Construct the 4-current J^{μ} for this situation. Then, by explicit Lorentz transformation, find the 4-current in the frame of a starship moving with respect to that of part a) with velocity $\mathbf{v}$ along the wire.

   (c) (2 pts.) By explicit Lorentz transformation, find the $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$ fields observed by the pilot of the starship. Express the result in terms of the current $\bar{I}$

observed by the pilot. Explain the physical meaning of your answers for the fields.

(d) (2 pts.) Is it possible to find a frame where **E**=0 for this system? Why or why not?

6. A t-dependent voltage $V(t) = V_0 \cos(\omega t)$ is applied to a capacitor, which consists of two concentric conducting spheres of radii a and b ($a < b$).



The space in between the spheres is filled with two spherical shells made of *different* insulators, so that

$$\epsilon = \begin{cases} \epsilon_1 & \text{for } a < r < c \\ \epsilon_2 & \text{for } c < r < b. \end{cases}$$

(a) (4 pts.) Find the capacitance C.

(b) (4 pts.) Find the displacement current (direction and magnitude) in terms of C.

(c) (2 pts.) Find the magnetic field produced by the displacement current.

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1 a)

Skin depth $= d = \frac{1}{\kappa}$

$$\kappa = \omega\sqrt{\frac{\varepsilon\mu_0}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right)^{1/2}$$

$$\kappa = \frac{5\times10^{15}}{\sqrt{2}}\underbrace{\frac{\sqrt{10}}{3\times10^{8}}}_{c}\left(\sqrt{1+(1.13)^2}-1\right)^{1/2}$$

$$= 2.66\times10^{7}\ m^{-1}$$

$$d = 1/\kappa = 3.76\times10^{-8}\ m = 37.6\,nm$$

index of refraction $n = \frac{kc}{\omega}$ ← real part of $\tilde{k}$

$$k = \omega\sqrt{\frac{\varepsilon\mu_0}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right)^{1/2}$$

$$= 3.73\times10^{7}\left(\sqrt{1+(1.13)^2}+1\right)^{1/2}$$

$$= 5.9\times10^{7}\ m^{-1}$$

$$n = \frac{ck}{\omega} = 3.54 \quad \text{for this frequency}$$

b) $\frac{\sigma}{\varepsilon\omega} = \frac{5\times10^{5}}{(8.85\times10^{-11})(5\times10^{15})} = 1.13$

Reflection coefficient $R = \left|\dfrac{\tilde{n}-1}{\tilde{n}+1}\right|$

$$\tilde{n} = \frac{c}{\omega}(k + i\kappa) \equiv n + i\frac{c}{\omega}\kappa$$

$$R = \frac{(n-1)^2 + \left(\frac{c\kappa}{\omega}\right)^2}{(n+1)^2 + \left(\frac{c\kappa}{\omega}\right)^2} \qquad \frac{c\kappa}{\omega} = \frac{(3\times10^8)(2.66\times10^7)}{(5\times10^{15})}$$

$$= 1.60$$

$$= \frac{2.54^2 + 1.6^2}{4.54^2 + 1.6^2} = 0.389$$

b) Expand for small $\dfrac{\sigma}{\varepsilon\omega}$:

$$k = \omega\sqrt{\frac{\varepsilon\mu_0}{2}}\sqrt{2} = \omega\sqrt{\varepsilon\mu_0}$$

$$\kappa = \omega\sqrt{\frac{\varepsilon\mu_0}{2}}\left(1 + \frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2 - 1\right)^{1/2}$$

$$= \omega\frac{1}{2}\sqrt{\varepsilon\mu_0}\left(\frac{\sigma}{\varepsilon\omega}\right) = \frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon}}$$

$$\tilde{k} \cong \omega\sqrt{\varepsilon\mu_0} + i\frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon}}$$

$$R = \left|\frac{\tilde{n}-1}{\tilde{n}+1}\right| = \left|\frac{\frac{c}{\omega}\tilde{k}-1}{\frac{c}{\omega}\tilde{k}+1}\right| = \left|\frac{c\sqrt{\varepsilon\mu_0} + \frac{ic\sigma}{2\omega}\sqrt{\frac{\mu_0}{\varepsilon}} - 1}{c\sqrt{\varepsilon\mu_0} + \frac{ic\sigma}{2\omega}\sqrt{\frac{\mu_0}{\varepsilon}} + 1}\right|$$

$$= \frac{\left(c\sqrt{\varepsilon\mu_0}-1\right)^2 + \left(\frac{c\sigma}{2\omega}\sqrt{\frac{\mu_0}{\varepsilon}}\right)^2}{\left(c\sqrt{\varepsilon\mu_0}+1\right)^2 + \left(\frac{\sigma c}{2\omega}\sqrt{\frac{\mu_0}{\varepsilon}}\right)^2}$$

2 Gauge transforms

a)
$$\vec{A}' = \vec{A} + \nabla\Lambda \qquad 1)$$

$$V' = V - \frac{\partial\Lambda}{\partial t} \qquad 2)$$

Eq. 2) can always be solved with $V'=0$ by choosing

$$\Lambda = \int_0^t V(t')\,dt'$$

but this doesn't work for 1) since there are 3 components. Easiest way to prove this is to note that if $\vec{A}'=0$, then $\vec{B}' = \nabla\times\vec{A}' = 0$ too. But in general

$$\nabla\times\left(\vec{A}+\nabla\Lambda\right) = \vec{\nabla}\times\vec{A} = \vec{B}.$$

So only soln $\vec{A}'=0$ is if $\vec{B}=0$.

b) Coulomb gauge is $\vec{\nabla}\cdot\vec{A}=0$

so yes, this works

Lorentz gauge $\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0 \frac{\partial V}{\partial t} = 0$

Since $\vec{A} = 0$, $V \neq V(t)$ OK

c) Gauge fctn $\Lambda = \frac{qt}{4\pi\varepsilon_0 r}$

$$V' = V - \frac{\partial\Lambda}{\partial t} = 0$$

$$\vec{A}' = \vec{A} + \nabla\Lambda = \frac{qt}{4\pi\varepsilon_0} \underbrace{\nabla\frac{1}{r}}_{\frac{-\hat{r}}{r^2}}$$

$$\vec{A}' = \frac{-qt\hat{r}}{4\pi\varepsilon_0 r^2}$$

Lorentz gauge $\vec{\nabla}\cdot\vec{A}' = \frac{-qt}{4\pi\varepsilon_0}\vec{\nabla}\left(\frac{\hat{r}}{r^2}\right)$

$$= \frac{-qt}{\varepsilon_0}\delta^3(\vec{r}) \neq 0$$

Neither Coulomb nor Lorentz gauge is satisfied at $\vec{r} = 0$

Singular gauge transform.

d) Can always find gauge $V' = 0$, $V = \frac{\partial\Lambda}{\partial t}$ "pure gauge"

$\rightarrow \Lambda(\vec{r},t) = \int^t dt'\, V(t')$

But cannot do same thing for $\vec{A}$ always.

If $\vec{A}' = 0$ then $\vec{B} = \nabla\times\vec{A}' = 0$ gauge-invariant

Therefore if $\vec{B} \neq 0$ $\vec{A}$ cannot be gauged away.

5

3 a) $F = m\ddot{x} = \underbrace{-kx}_{\text{binding}} \underbrace{- m\gamma\dot{x}}_{\text{damping}} + \underbrace{qE_0 e^{-i\omega t}}_{\text{driving}}$

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{-eE_0}{m} e^{-i\omega t}}$$

$\omega_0 = \sqrt{\frac{k}{m}}$

b) $\tilde{x} = \tilde{x}_0 e^{-i\omega t}$

$$-\omega^2 \tilde{x}_0 - i\omega\gamma\tilde{x}_0 + \omega_0^2 \tilde{x}_0 = \frac{-eE_0}{m}$$

$$\Rightarrow \tilde{x}_0 = \frac{eE_0/m}{\omega^2 + i\omega\gamma - \omega_0^2}$$

$$\boxed{\tilde{x}(t) = \frac{eE_0/m}{\omega^2 + i\omega\gamma - \omega_0^2} e^{-i\omega t}}$$

$$\tilde{a} = \ddot{\tilde{x}} = \frac{-e\omega^2 E_0/m}{\omega^2 + i\omega\gamma - \omega_0^2} e^{-i\omega t}$$

c) $\tilde{\vec{p}}(t) = (-e)\tilde{x}(t)$; Physical $\vec{p} = \mathrm{Re}\,\tilde{p}$

d) Power radiated given by Larmor formula:

$$P = \frac{\mu_0 e^2}{6\pi c} a^2$$

where $a = \mathrm{Re}\,\tilde{a}$

$$= \frac{\omega^2 eE_0/m}{(\omega^2-\omega_0^2)^2 + \omega^2\gamma^2}\left[\sin\omega t\,\omega\gamma + (\omega_0^2-\omega^2)\cos\omega t\right]$$

Average $\langle a^2 \rangle$ over complete cycle

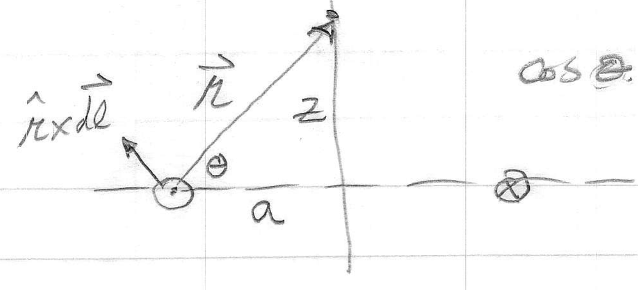
$$\langle a^2 \rangle = \left(\frac{\omega^2 e E_0/m}{(\omega^2-\omega_0^2)^2+\omega^2\gamma^2} \right)^2 \left[\omega^2\gamma^2 \langle \sin^2 \omega t \rangle + (\omega_0^2-\omega^2)^2 \langle \cos^2 \omega t \rangle \right]$$

$$= \left(\frac{\omega^2 e E_0/m}{(\quad)^2+\omega^2\gamma^2} \right)^2 \cdot \frac{1}{2} \left[(\quad)^2 + \gamma^2\omega^2 \right]$$

$$= \frac{\omega^4 e^2 E_0^2/2m^2}{\left[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2 \right]}$$

$$\langle P \rangle = \frac{\mu_0}{12\pi c} \frac{(e^4 E_0^2/m^2)\,\omega^4}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$

4 (a)



$\cos\theta = \dfrac{a}{(z^2+b^2)^{1/2}}$

z-component

Biot-Savart law

$$\vec{B}(\vec{r}=z\hat{z}) = \frac{\mu_0}{4\pi} I_a \int \frac{d\vec{l}\times\hat{r}}{r^2} = \frac{\mu_0 a I_a}{2(z^2+b^2)} \cos\theta\, \hat{z}$$

$$= \frac{\mu_0 a^2 I_a}{2(z^2+b^2)^{3/2}}\hat{z}$$

(b) If $b \ll a$ we may assume that the field produced by the circular loop is constant over the square one. Hence

$$\Phi_b = \pi b^2 \cdot \frac{\mu_0 a^2 I_a}{2(z^2+b^2)^{3/2}}$$

Def. of mutual inductance $\Phi_b = M_{ba} I_a$

$$\Rightarrow \quad M_{21} = \frac{\pi \mu_0 a^2 b^2}{2(z^2+b^2)^{3/2}}$$

Now z is height of b above a

(c) $$\mathcal{E}_b = \frac{-d\Phi_b}{dt} = -M_{ba}\frac{dI_a}{dt} = +M_{ba} I_0 \omega \sin\omega t$$

$$= M_{ba} I_0 \omega \sin\omega t \equiv \oint \vec{E}_{ind}\cdot d\vec{l}$$

$$\Rightarrow \vec{E}_{ind} = \frac{M_{ba} I_0 \omega \sin\omega t}{2\pi b}\hat{\theta}$$

5.

8

(Line charge) Consider an infinitely long, straight, infinitesimally thin line charge which has charge density per unit length λ in its own rest frame.

(a) (2pts.) Write down the $\mathbf{E}$ and $\mathbf{B}$ fields in this frame, and construct the electromagnetic field tensor $F^{\mu\nu}$.

line chg. at rest $\Rightarrow$ $\vec{B}=0$, $\vec{E}=\frac{2\lambda}{4\pi\varepsilon_0 s}\hat{s}$

$$\vec{E}=\frac{2\lambda}{4\pi\varepsilon_0}\frac{\hat{s}}{s}=\frac{2\lambda}{4\pi\varepsilon_0}\frac{1}{\sqrt{x^2+y^2}}(\cos\phi\,\hat{x}+\sin\phi\,\hat{y}) \equiv E_x\hat{x}+E_y\hat{y}$$

$$F^{\mu\nu}=\begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & E_x/c & E_y/c & 0 \\ -E_x/c & 0 & 0 & 0 \\ -E_y/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4-current $J^\mu=(c\rho,\vec{J})$ (b) (2 pts.) Construct the 4-current J^μ for this situation. Then, by explicit Lorentz transformation, find the 4-current in the frame of a starship moving with respect to that of part a) with velocity $\mathbf{v}$ along the wire. z-axis is wire!

Rest Frame of line charge $\vec{J}=0$ $\rho(\vec{r})=\lambda\,\delta^{(2)}(\vec{s})$

$$\bar{J}^\mu=\begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}\begin{bmatrix} \rho c \\ 0 \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} \gamma\rho c \\ 0 \\ 0 \\ -\beta\gamma\rho c \end{bmatrix} \qquad \gamma=\frac{1}{\sqrt{1-\beta^2}} \quad \beta=\frac{v}{c}$$

So... in starship frame $\bar{\rho}=\gamma\rho c$ $\bar{\vec{J}}=-\beta\gamma\rho c\,\hat{z}$

(c) (2 pts.) By explicit Lorentz transformation, find the $\bar{\mathbf{E}}$ and $\bar{\mathbf{B}}$ fields observed by the pilot of the starship. Express the result in terms of the current $\bar{I}$ observed by the pilot. Explain the physical meaning of your answers for the fields.

$$\bar{F}=\begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}\begin{bmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & 0 \\ -\frac{E_x}{c} & 0 & 0 & 0 \\ -\frac{E_y}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix}=\begin{bmatrix} 0 & \frac{\gamma E_x}{c} & \frac{\gamma E_y}{c} & 0 \\ -\frac{\gamma E_x}{c} & 0 & 0 & \frac{\beta\gamma E_x}{c} \\ -\frac{\gamma E_y}{c} & 0 & 0 & \frac{\beta\gamma E_y}{c} \\ 0 & -\frac{\beta\gamma E_x}{c} & -\frac{\beta\gamma E_y}{c} & 0 \end{bmatrix}$$

So in moving frame we have larger E-field (from contracted charge density) $\gamma\vec{E}$

and magnetic field $\bar{B}_z=0$ $\bar{B}_y=-\frac{\gamma\beta E_x}{c}$ $\bar{B}_x=\frac{\beta\gamma E_y}{c}$

Current in this frame $\vec{J}=-\beta\gamma\rho c\hat{z}$ (to left!) $\equiv \bar{I}\,\delta^{(2)}(\vec{s})\,\hat{z}$

where $\bar{I}=-\beta\gamma\lambda c$. And $\vec{B}=-\frac{\beta\gamma}{c}\hat{z}\times\vec{E}=-\frac{\beta\gamma}{c}\frac{2\lambda}{4\pi\varepsilon_0 s}\hat{\phi}=\frac{\mu_0\bar{I}}{2\pi s}\hat{\phi}$

just field of current $\bar{I}$ in infinite wire

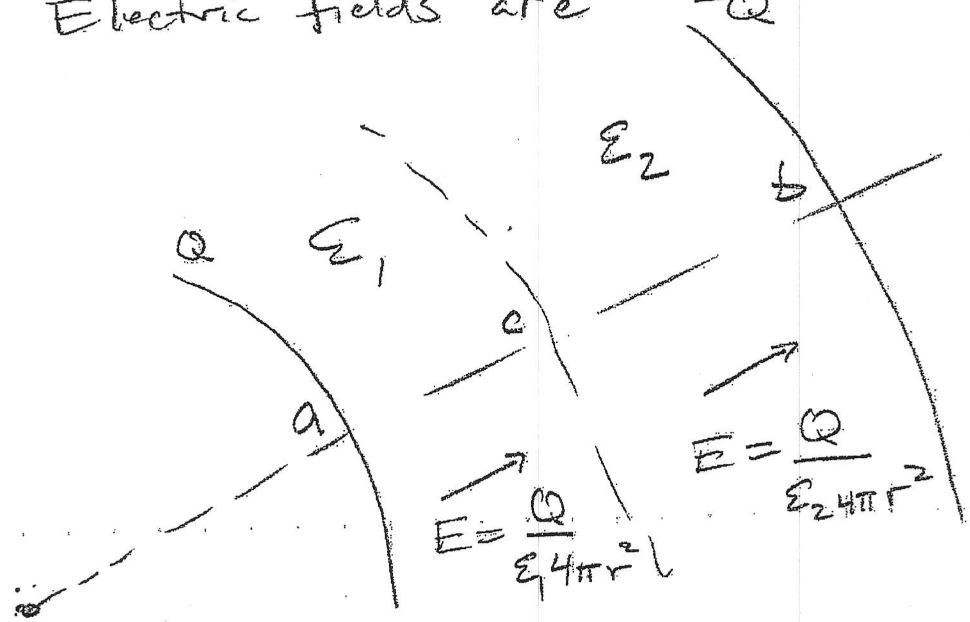
(d) (1.5 pts.) Is it possible to find a frame where E=0 for this system? Why or why not?

Not possible. $E^2 - c^2B^2$ is <u>Lorentz invariant</u>

In lab frame $B=0 \Rightarrow E^2 - c^2B^2 > 0$, must stay > 0 in all frames. $\Rightarrow E \neq 0$.

6 (a) Capacitance: put Q on inner conductor

Electric fields are



$$V = V(a) - V(b) = -\int_b^a \vec{E}\cdot d\vec{r} = \int_a^b E\,dr$$

$$= \frac{Q}{4\pi\varepsilon_1}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{Q}{4\pi\varepsilon_2}\left(\frac{1}{c} - \frac{1}{b}\right)$$

c

$Q = CV \Rightarrow$
$$\boxed{C = \frac{1}{\frac{1}{4\pi\varepsilon_1}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{1}{4\pi\varepsilon_2}\left(\frac{1}{c} - \frac{1}{b}\right)}}$$

c

(b) Displacement current density $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$

Displacement current $I_d = \int \vec{J}_d \cdot d\vec{a} = J_d 4\pi r^2$

$\vec{D}$ is displacement field due to <u>free</u> <u>charges</u>:

$$\vec{D} = \frac{Q}{4\pi r^2} \Rightarrow J_d = \frac{\dot{Q}}{4\pi r^2} = \frac{C}{4\pi r^2}\dot{V}$$

$$\Rightarrow J_d = -\frac{C}{4\pi r^2} V_0 \omega \sin \omega t$$

$$\boxed{I_d = -C V_0 \omega \sin \omega t}$$

(c) Magnetic field due to displacement current

$= \underline{\underline{0}}$ by symmetry

since $\vec{J}_d$ is <u>radial</u> $\propto \hat{r}$