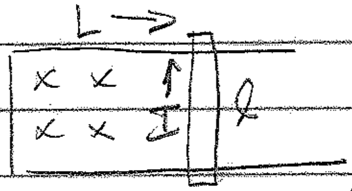
Exam 1 Solutions

1) Sliding rail.



$v = \frac{dL}{dt}$ $\quad \longrightarrow \hat{x}$

a) $$-\frac{d\left[B(L\ell)\right]}{dt} = -B\ell v = \int \vec{E}\cdot d\vec{\ell} = E\ell = \mathcal{E}$$

resistance $R = \left(\frac{1}{\sigma}\right)\frac{\ell}{A}$ $\qquad \mathcal{E} = IR$

$$I = \frac{\mathcal{E}}{R} = \boxed{\frac{-B\ell v}{R} = -Bv\sigma A \text{ (counter-clockwise)}}$$

b) $$\vec{F} = I\vec{\ell}\times\vec{B} = \boxed{I\ell B = \frac{B^2\ell^2 v}{R} = B^2\ell v\sigma A \text{ to left! } (-\hat{x})}$$

c) $$F = -\frac{B^2\ell^2}{R}v = m\frac{dv}{dt} \quad \text{(Newton)}$$

solution $v = v_0 e^{-at}$

$$-\frac{B^2\ell^2\cancel{v}}{R} = m(-a)\cancel{v}$$

$$a = \frac{B^2\ell^2}{mR} > 0$$

$$\boxed{v = v_0 e^{-B^2\ell^2 t/(mR)}}$$

d) $$\Delta x = \int_0^\infty dt\cdot v(t) = v_0\int_0^\infty dt\; e^{-at}$$

$$= v_0\left(\frac{-1}{a}\right)(0-1) = \boxed{\frac{v_0 mR}{B^2\ell^2} = \frac{v_0 m}{B^2\sigma A\ell}}$$

e) Joule power dissipated $= I^2 R$

$$= \frac{B^2 \ell^2 v^2}{R}$$

Total energy dissipated $\int_0^\infty dt\, I^2(t) R$

$$= B^2 \ell^2 R^{-1} \int_0^\infty dt\, v_0^2 e^{-2at}$$

$$= -\frac{B^2 \ell^2 v_0^2}{2aR}(0-1) = \frac{B^2 \ell^2 v_0^2}{2R\cdot(B^2\ell^2/(mR))}$$

$$= \tfrac{1}{2} m v_0^2 \quad \checkmark$$

2) B of wires: top: $\frac{\mu_0 I_W}{2\pi s}\hat{\phi}$

bottom: $\frac{\mu_0 I_W}{2\pi(a+b+c-s)}\hat{\phi}$

a) total flux $= \Phi_{top} + \Phi_{bot}$

$$\Phi_{top} = d\mu_0 I_W \int_a^{a+b} \frac{ds}{2\pi s} = d\mu_0 I \ln\frac{a+b}{a}$$

$$\Phi_{bot} = d\mu_0 I_W \int_a^{a+b} \frac{ds}{2\pi(a+b+c-s)}$$

$$= \frac{d\mu_0 I_W}{2\pi}\left(-\ln\frac{c}{b+c}\right) = \frac{d\mu_0 I_W}{2\pi}\ln\frac{b+c}{c}$$

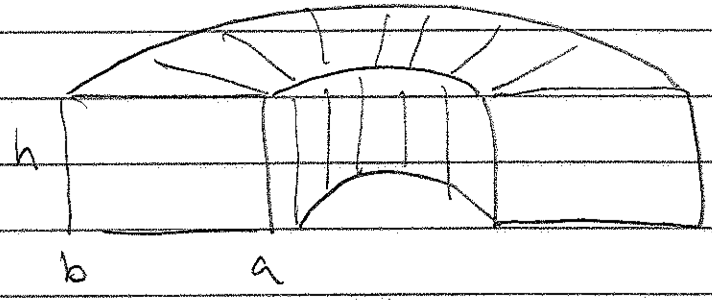
3

$$\Phi_{top} + \Phi_{bot} = \boxed{\frac{d\mu_0 I_w}{2\pi} \ln \frac{(a+b)(b+c)}{ac}}$$

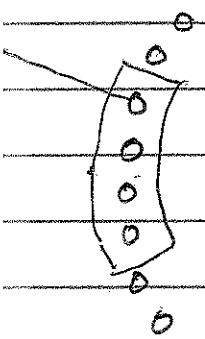
b) $\Phi_2 = M_{21} I_1$

$$M_{21} = \frac{\mu_0 d}{2\pi} \ln \frac{(a+b)(b+c)}{ac}$$

c) $\vec{B}$ field inside toroid is



$$\vec{B} = \begin{cases} \frac{\mu_0 N I}{2\pi s} \hat{\phi} & b > s > a \\ 0 & \text{otherwise} \end{cases}$$



(see Griffiths example 5.10 or take Amperian loop:

$$B \cdot 2\pi s = \mu_0 N I \quad \Rightarrow B = \frac{\mu_0 N I}{2\pi s} \Big)$$

Flux for single rectangular loop:

$$\Phi_{loop} = \frac{\mu_0 I N h}{2\pi} \int_a^b \frac{1}{s} ds = \frac{\mu_0 I N h}{2\pi} \ln \frac{b}{a}$$

for N total loops

$$\Phi_{tot} = \frac{\mu_0 I N^2 h}{2\pi} \ln \frac{b}{a}$$

$$\boxed{\text{self-inductance} \quad L = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}}$$

as in class

3) solenoid

a) $$\vec{E} = \begin{cases} -\frac{\sigma}{\varepsilon_0}\hat{y} & \text{between planes } -\frac{d}{2} \le y \le \frac{d}{2} \\ 0 & \text{elsewhere} \end{cases}$$

$$\vec{B} = \begin{cases} \mu_0 n I \hat{z} & s^2 = x^2 + y^2 \le R^2 \\ 0 & \text{otherwise} \end{cases}$$

b) $$T_{ij} = \varepsilon_0\left(E_iE_j - \frac{1}{2}E^2\delta_{ij}\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$$

N.B.
a) inside solenoid
$E, B \neq 0$
outside solenoid but between plates
set $B = 0$
outside everything
$\Leftrightarrow$ $T = 0$

$$T_{xx} = -\frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = -u$$

$$T_{yy} = \frac{1}{2}\varepsilon_0 E^2 - \frac{1}{2\mu_0}B^2$$

$$T_{zz} = -\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2$$

$$T_{ij} = 0 \quad \text{if } i \neq j$$

c) Momentum density $\vec{p}_{em} = \mu_0 \varepsilon_0 \overbrace{S}^{\frac{1}{\mu_0}\vec{E}\times\vec{B}}$

$$= \varepsilon_0\left(-\frac{\sigma}{\varepsilon_0}\mu_0 n I \hat{x}\right) \quad \text{inside solenoid}$$

$$= \mu_0 I \sigma n \hat{x}$$

4. EM wave

a) Short answer

i) EM wave is transverse since amplitudes of $\vec{E}, \vec{B} \perp$ to direction of propagation $\vec{k}$.

ii) False. Wave need not be $\propto \cos(kz-\omega t)$ or $\sin(kz-\omega t)$ or even a combination thereof. A general wavelike soln is of the form $\vec{E}, \vec{B} \propto (z \pm vt)$. A wave pulse consists of many frequencies ω.

iii) False. EM waves can propagate in a vacuum.

iv) In a circularly polarized wave the amplitudes of $\vec{E}$ and $\vec{B}$ rotate in the plane $\perp$ to propagation
e.g. $\vec{E}(x,t) = \left(E_0^x \cos\omega t\, \hat{x} + E_0^y \sin\omega t\, \hat{y}\right) \cdot \cos kx$

b) Maxwell Eqns in vacuum

i) $\nabla \cdot \vec{E} = 0$ iii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
ii) $\nabla \cdot \vec{B} = 0$ iv) $\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

apply $\vec{\nabla} \times$ to iv and use $\vec{\nabla} \times \vec{\nabla} \times \vec{B}$
$= \nabla(\cancel{\nabla \cdot \vec{B}}) - \nabla^2 \vec{B} = -\nabla^2 B$

$$\text{so} \quad -\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \vec{\nabla} \times \vec{E}$$

$$= \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

ME iii)

$$\Rightarrow \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

wave eqn

of general form $\nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$

$$\Rightarrow c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Same for $\vec{E}$!

c) Assume $\vec{E} = \vec{E}_0 e^{i(kz-\omega t)}$

$\vec{B} = \vec{B}_0 e^{i(kz-\omega t)}$

Faraday: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_0^x e^{i(kz-\omega t)} & E_0^y e^{i(kz-\omega t)} & 0 \end{vmatrix}$$

$$= -\hat{x} E_0^y ik e^{i(\,)} + \hat{y} E_0^x ik e^{i(\,)}$$

$$= \frac{-\partial \vec{B}}{\partial t} = \hat{x} B_0^x i\omega e^{i(\,)} + \hat{y} B_0^y i\omega e^{i(\,)}$$

$$\Rightarrow B_0^x \omega = -E_0^y k \;;\; B_0^y \omega = E_0^x k$$

$$\vec{B}_0 \cdot \vec{E}_0 = B_0^x E_0^x + B_0^y E_0^y$$

$$= \left(B_0^x B_0^y - B_0^y B_0^x\right) \frac{\omega}{k} = 0 \quad \checkmark$$