11 Complex numbers

Read: Boas Ch. 2

Represent an arb. complex number $z \in \mathbb{C}$ in one of two ways:

$$\begin{aligned} z &= x + iy \;\; ; \qquad x, y \in \mathbb{R} \qquad \text{“rectangular” or ”Cartesian” form”} \\ z &= re^{i\theta} \quad ; \qquad r, \theta \in \mathbb{R} \qquad \text{“polar” form.} \end{aligned} \tag{1}$$

Here i is $\sqrt{-1}$, engineers call it j (ychh! The height of bad taste.). If $z_1 = z_2$, both real and imaginary parts are equal, $x_1 = x_2$ and $y_1 = y_2$. This implies of course that “0” $\in \mathbb{C}$” is the complex number with both real *and* imaginary parts 0.

Defs.: The *complex conjugate* of $z = x + iy$ is defined to be $z^* = x - iy$ (Boas sometimes calls the same thing $\overline{z}$). Note $zz^* = x^2 + y^2$, a real number ≥ 0. The *modulus* or *magnitude* of z is then defined to be $|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}$, has obvious analogies to the distance function in Euclidean space. Some useful relations you can easily work out are

$$\mathrm{Re}z = \frac{z + z^*}{2} \quad ; \quad \mathrm{Im}z = \frac{z - z^*}{2i} \quad ; \quad (z_1 z_2)^* = z_1^* z_2^*. \tag{2}$$

You should practice simplifying complex numbers which are not given explicitly in the form $x + iy$. For example,

$$\frac{\sqrt{2} + i}{1 - i} = \frac{\sqrt{2} + i}{1 - i}\left(\frac{1 + i}{1 + i}\right) = \frac{(\sqrt{2} + i)(1 + i)}{2} = \frac{\sqrt{2} - 1}{2} + i\frac{\sqrt{2} + 1}{2} \tag{3}$$

Polar form:

Just as all real numbers can be represented as points on a line, complex numbers can be represented and manipulated as points in a 2D space spanned by the real and imaginary parts x and y.

Euler theorem

Recall the expansion of the exponential function $e^x = \sum_n x^n/n!$, which converges for arbitrary size of x. Let's define a complex exponential e^z in the same way, and choose in particular $z = i\theta$. Then

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n (\theta)^{2n}}{(2n)!}}_{\text{even terms, } \cos\theta} + i \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n (\theta)^{2n+1}}{(2n+1)!}}_{\text{odd terms, } \sin\theta} \\ &= \cos\theta + i\sin\theta, \end{aligned} \tag{4}$$

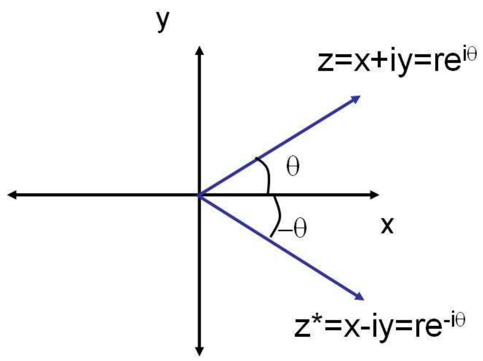

Figure 1: Representation of a complex number z and its conjugate z^*.

where I used $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc. and identified the series for sin and cos of a real variable! Then going back and forth between rectangular and polar form is as easy as

$$z = x + iy = r[\frac{x}{r} + i\frac{y}{r}] = r[\cos\theta + i\sin\theta] = re^{i\theta} \tag{5}$$

Note

$$\cos\theta = \mathrm{Re}\, e^{i\theta} = \frac{e^{i\theta} + e^{i\theta}}{2} \quad ; \quad \sin\theta = \mathrm{Im}\, e^{i\theta} = \frac{e^{i\theta} - e^{i\theta}}{2i}. \tag{6}$$

Now we can see an interesting relationship between these functions and their hyperbolic analogs,

$$\cosh\theta = \frac{e^{\theta} + e^{\theta}}{2} \quad ; \quad \sinh\theta = \frac{e^{\theta} - e^{\theta}}{2}. \tag{7}$$

11.1 Roots and powers of complex variable

Powers:

$$z = a + ib = re^{i\theta} \quad ; \quad z^n = r^n e^{in\theta} \tag{8}$$

<u>Ex.:</u> $(1+i)^3 = (\sqrt{2}e^{i\pi/4})^4 = 4e^{i\pi} = -4$

Roots require a bit more care since they are fractional powers:

$$\sqrt[n]{z} = \sqrt[n]{re^{i\theta}} = r^{1/n}e^{i(\theta+2m\pi)/n} \qquad m = 0, \pm1, \pm2, ... \tag{9}$$

This seems a little paradoxical. Adding $2m\pi$ to the argument of the exponential doesn't change z. Nevertheless it has different nth roots according to how you choose n. Any of $e^{i\theta/n} \cdot e^{i2\pi m/n}$ are perfectly acceptable nth roots of z. These are called the *branches* of the complex root function.

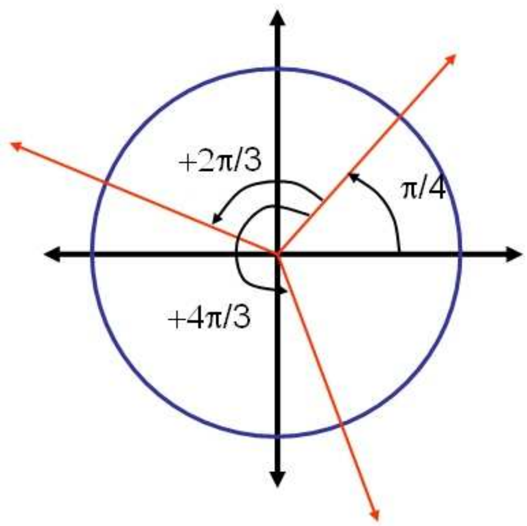

Figure 2: Roots of $\sqrt[3]{-2+2i}$. Note there are precisely 3 distinct roots in the complex plane.

<u>Ex.:</u>

$$\begin{aligned}
\sqrt[3]{-2+2i} &= (\sqrt{8}e^{i\frac{3\pi}{4}})^{1/3} = 8^{1/6}e^{i(\frac{\pi}{4}+\frac{2\pi m}{3})} \qquad m=0,1,2 && (10)\\
m=0: &\quad \sqrt{2}e^{i\pi/4} && (11)\\
m=1: &\quad \sqrt{2}e^{i(\frac{\pi}{4}+\frac{2\pi}{3})} && (12)\\
m=2: &\quad \sqrt{2}e^{i(\frac{\pi}{4}+\frac{4\pi}{3})} && (13)\\
m=3: &\quad \text{same as } m=1 && (14)\\
m=4: &\quad \text{same as } m=2 && (15)\\
\vdots & && (16)
\end{aligned}$$

Q: How about $\sqrt[3]{1}$?

11.2 Complex power series, functions of complex variable

General: $\sum_n a_n z^n$

<u>Ex.:</u>

$$1 - z + z^2/2 - z^3/3 + \dots \tag{17}$$

Ratio test for absolute convergence (remember before we require that ratio of successive terms be less than 1, if limit exists):

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}z}{a_n}\right| : \qquad \text{converges if } \rho < 1 \tag{18}$$

so series in our example above converges if $|z| < 1$. This determines a disk of radius ρ in complex plane, the "disk of convergence". The exponential series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{19}$$

has an infinite radius of convergence since

$$\rho = \left| \frac{n! z}{(n+1)!} \right| \to \frac{|z|}{n+1} \to 0 \tag{20}$$

for any z. Similarly we define other complex functions by their power series, e.g.

$$\sinh z \equiv \frac{e^z - e^{-z}}{2} \quad ; \quad \cosh z \equiv \frac{e^z + e^{-z}}{2} \tag{21}$$

$$\sin z \equiv \frac{e^{iz} - e^{-iz}}{2i} \quad ; \quad \cos z \equiv \frac{e^{iz} + e^{-iz}}{2}. \tag{22}$$

Note $\cos^2 z + \sin^2 z = 1$ remains valid for complex z, as do all trig relations you know for real z. Now we see relations between hyperbolic and ordinary trig functions:

$$\sin iz = i \sinh z \quad ; \quad \sinh iz = i \sin z \tag{23}$$

$$\cos iz = \cosh z \quad ; \quad \cosh iz = \cos z \tag{24}$$

Log function:

$$\ln(1+z) \equiv \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} z^n, \tag{25}$$

and we have $e^{\ln z} = z$ as for real nos. However, consider $z = re^{i\theta}$:

$$\ln z = \ln(re^{i\theta}) = \mathrm{Ln}\, r + i(\theta + 2\pi n), \tag{26}$$

i.e. infinitely many values. The Ln written with large L is the natural log for real argument. $n = 0$ gives what is called the "principal value" of the log function:

$$\ln z = \mathrm{Ln}\, r + i\theta, \tag{27}$$

but don't forget that the other values, or "branches" are also allowed. Ex:

$$\ln(-1) = \ln e^{i\pi} = \mathrm{Ln}\, 1 + i(\pi + 2n\pi) = \pm i\pi, \pm 3i\pi, \ldots, \tag{28}$$

but the principal value is $i\pi$.

Note:

$$z^{1/n} = e^{\frac{1}{n}\ln z} = e^{\frac{1}{n}[\mathrm{Ln}\, r + i(\theta + 2m\pi)]} = r^{1/n} e^{i(\frac{\theta}{n} + \frac{2m\pi}{n})} \tag{29}$$

11.3 Complex exponents and roots

Ex.:

$$(2i)^{1+i} = e^{(1+i)\ln 2i} = e^{(1+i)[\ln 2 + \ln i]} \tag{30}$$

Now i itself may be written $i = e^{i(\frac{\pi}{2} \pm 2\pi n)}$, so $\ln i = i(\frac{\pi}{2} \pm 2\pi n) \equiv i\alpha$. So

$$e^{(1+i)[\ln 2+\ln i]} = e^{\ln 2 - \alpha + i(\ln 2+\alpha)} = 2e^{-\alpha} e^{i(\ln 2+\alpha)}. \tag{31}$$

11.4 Applications of complex numbers in physics

Here I just give a few random examples where these concepts are useful:

- Adding harmonic waves with fixed phase. Suppose you want to know what the sum of a bunch of waves is (I'll leave out the physical dimensions, but you can imagine electric field, height of a water wave, ... it doesn't matter much).

  $$S = \sin x_0 + \sin(x + x_0) + \sin(2x + x_0) + \sin(3x + x_0) \ldots \sin(Nx + x_0) \tag{32}$$

  This is hard to get a closed form for because there are many terms and one can't combine terms without using trig identities, which are messy. How about this, let's write $S = \mathrm{Im} S_1$, where

  $$\begin{aligned} S_1 &= e^{ix_0} + e^{i(x+x_0)} + e^{i(2x+x_0)} + \cdots + e^{i(Nx+x_0)} \\ &= e^{ix_0}[1 + z + z^2 + z^3 + \cdots + z^N], \end{aligned} \tag{33}$$

  where $z = e^{ix}$. The terms in square brackets are just a geometric series, with sum $(1 - z^{N+1})/(1 - z)$. So we could obtain a closed form expression

  $$S = \mathrm{Im}\left(e^{ix_0}\frac{1 - e^{iNx}}{1 - e^{ix}}\right). \tag{34}$$

- Circular motion in complex plane and its uses. If we take a function of time $z(t) = re^{i\omega t}$, it is clear that z is following a circle of radius r in the complex plane, starting at (r,0) at t=0, and going counterclockwise. The speed of "rotation" is $|dz/dt| = \omega r$, etc. So we can use $e^{i\omega t}$ to represent an object going in a circle at angular frequency ω, or $\mathrm{Re}e^{i\omega t}$ to represent an object oscillating back and forth with position $x = r\cos\omega t$, etc. This has important technical uses in physics. If something is oscillating with time with sinusoidal or cosinusoidal form, we represent the "signal" as the real or imaginary part of a complex exponential. Often the resulting equations are much simpler to solve due to the nice mathematical properties of exponentials, and we can always take the real part of the solution at the end of the calculation to obtain the physical observable desired. Here's an example.

- Consider a series LRC circuit as shown. The voltage drop across the resistor is IR (where I is the current through the resistor), across the capacitor is Q/C (where Q is the charge on the capacitor), and across the inductor is $(dI/dt)L$ (due to Faraday's law). Since the current leading to the capacitor has nowhere else to go, it must be that $I = dQ/dt$. We can therefore express Kirchoff's law for the sum of the voltage drops around the circuit (including the voltage source $V(t)$ equaling zero as

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_0 \cos \omega t, \tag{35}$$

We assumed a cosinusoidal form for the driving voltage for simplicity, but this is not essential.

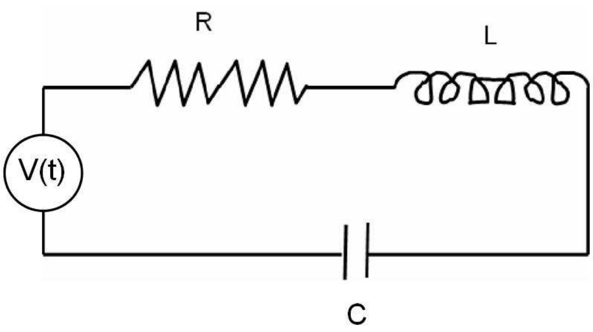

Figure 3: Series circuit with inductor (L), resistor (R), and capacitor (C).

You probably solved a circuit like this in elementary E&M, but let's review. First consider a simple case with no capacitance or inductance, so only a loop with the single element R. Then from Ohm's law we know that the current and the voltage are proportional,

$$\begin{aligned} I &= V/R \qquad (36) \\ &= \frac{V_0}{R}\cos \omega t. \qquad (37) \end{aligned}$$

Note both the *current* and the voltage have the same cosine form. Second, consider the case $R = 0$, i.e. only L and C present. Guess[1] $Q = Q_0 \cos \omega t$, and substitute into (35). We get

$$L(-\omega^2)Q_0 \cos \omega t + \frac{Q_0}{C}\cos \omega t = V_0 \cos \omega t \tag{38}$$

[1]note this way the differential equation is converted to an algebraic equation with parameters (Q_0 here) to be determined by substitution–very useful strategy.

which has the solution

$$Q_0 = \frac{V_0}{-\omega^2 L + C}. \tag{39}$$

Note in this case the *charge* and voltage have the cosine form.

General case. Now if all 3 circuit elements are present, $Q = Q_0 \cos \omega t$ is not a solution, and neither is $Q_0 \sin \omega t$. We can make a linear combination of the two and get the right answer, but it's tedious. Another way is to free our minds a bit, and imagine that the voltage is complex oscillatory, i.e. $V = V_0 e^{i\omega t}$. We will simply remember that the real voltage applied is the Re part of this expression, and use the fact that the mathematics becomes much simpler, *remembering always to take the Re part of all quantities at the end of the calculation.* So then let's make the ansatz $Q = Q_0 e^{i\omega t}$, and substitute into the differential equation. We obtain

$$[-\omega^2 L + i\omega R + \frac{1}{C}]Q_0 e^{i\omega t} = V_0 e^{i\omega t} \Rightarrow Q_0 = \frac{V_0}{-\omega^2 L + i\omega R + \frac{1}{C}}, \tag{40}$$

so we can write

$$Q(t) = \frac{V_0 e^{i\omega t}}{-\omega^2 L + i\omega R + \frac{1}{C}} \tag{41}$$

$$I(t) = \frac{dQ}{dt} = \frac{V_0 e^{i\omega t}}{i\omega L + R + \frac{1}{i\omega C}} \equiv \frac{V_0 e^{i\omega t}}{Z} = \frac{V(t)}{Z}, \tag{42}$$

so the current (the "complex current", not the physical one) looks just like our Ohm's law expression for the pure resistive case (R only), but with an "effective resistance" Z which we call the total impedance of the circuit. We notice that it consists of three terms $Z = Z_R + Z_L + Z_C$, one associated with each of the circuit elements

$$Z_R = R \ ; \quad Z_C = \frac{1}{i\omega C} \ ; \quad Z_L = i\omega L. \tag{43}$$

Now don't forget that the physical current is the real part of the result we obtained! So the physical current is (I'll be sloppy with notation and keep the same symbol):

$$I(t) = \mathrm{Re}\, \frac{V_0 e^{i\omega t}}{Z}. \tag{44}$$

It's easier to take the real part if we put the impedance in polar form:

$$Z = |Z| e^{i\phi}$$

$$|Z| = \sqrt{R^2 + (\omega L - 1/\omega C)^2} \quad ; \quad \phi = \tan^{-1} \frac{\omega L - 1/\omega C}{R}. \tag{45}$$

So

$$I(t) = \frac{V_0}{|Z|}\mathrm{Re}\, e^{i(\omega t - \phi)} = \frac{V_0}{|Z|}\cos(\omega t - \phi) \tag{46}$$

So the current lags the driving voltage by a phase ϕ which depends on the impedance of the circuit.

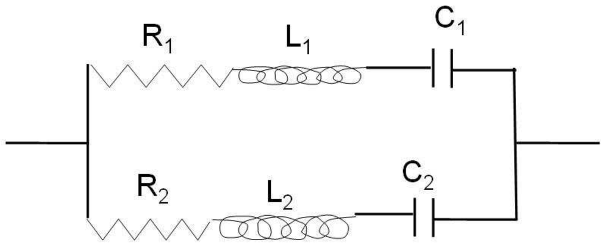

Figure 4: Parallel circuit with inductors (L_1 and L_2), resistors (R_1 and R_2), and capacitors (C_1 and C_2).

Now we could have guessed a form $I \propto \cos(\omega t + \phi)$, plugged it into (35) and solved for ϕ without all the complex number nonsense. But it gets progressively harder to make such guesses as the circuits become more complicated. By contrast, the complex voltage method is easily systematized. Consider a circuit element consisting of two impedence branches in parallel, as in Fig. 4. In the complex voltage approach, we note that the voltage across both legs is the same, and the current is the sum of the current in both legs. Therefore for the complex currents

$$I_1 = V/Z_1 \ \ ; \quad I_2 = V/Z_2 \qquad \Rightarrow I = I_1 + I_2 = V(\frac{1}{Z_1} + \frac{1}{Z_2}), \tag{47}$$

so parallel impedances add just like resistances in parallel,

$$\frac{1}{Z_{tot}} = \frac{1}{Z_1} + \frac{1}{Z_2} \qquad \text{parallel circuits.} \tag{48}$$

The impedance of each leg is $Z_L + Z_R + Z_C$ in the example shown, since the 3 elements are in series with each other as before. Looking at general series circuits it's also easy to show

$$Z_{tot} = Z_1 + Z_2 \qquad \text{series circuits.} \tag{49}$$