# 2 Differential calculus

## 2.1 Asymptotics of functions: blackbody radiation

One of the hardest things to teach students is how to have a qualitative feel for the important aspects of different functions. The first thing to always do when you are studying a function is consider its asymptotics, i.e. its behavior at 0 and $\infty$ and possibly near other $x$ where the function is singular.

Let's take the example of the blackbody radiation function $u(\lambda)$. This is the density of radiation per unit wavelength emitted by a blackbody (i.e. a perfectly absorbent body) in thermal equilibrium at temperature $T$,

$$u(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}d\lambda, \tag{1}$$

where $h$ is Planck's constant, $c$ the speed of light, $k$ Boltzmann's constant, and $\lambda$ the wavelength of light omitted. The easiest thing to think about is an object with a cavity inside, with radiation of all frequencies emitted by the blackbody filling the cavity. Actually, the *first* thing to do is to analyze the equation to make sure it's dimensionally correct. $h$ has the dimensions of angular momentum, i.e. $[h] = mL^2/t$. The quantity $hc$ therefore has dimensions $mL^3/t^2$, and the quantity $hc/\lambda$ dimensions $mL^2/t^2$, which has dimensions of an energy. Therefore the exponent in the denominator of Eq. 1 is indeed dimensionless, as it should be before we can raise it to a power! The factor which has dimensions, $hc(d\lambda)/\lambda^5$ then has dimensions $(mL^3/t^2)/L^4$, or $m/(Lt^2)$, which is the same dimensions as $[energy]/[volume]$, so that checks! The equation is therefore ok.

Our analysis of dimensions gives us hints about how to approach understanding the function. It contains a lot of physical quantities, but most of these are *constants*. We are interested in how $u(\lambda)$ depends on wavelength $\lambda$. Let's define a new *dimensionless* variable $x \equiv \lambda kT/(hc)$, which is proportional to $\lambda$. In terms of $x$, the formula reads

$$u(x) = 8\pi hc\left(\frac{kT}{hc}\right)^5 \left(\frac{1}{x}\right)^5 \frac{1}{e^{(1/x)}-1} \tag{2}$$

Let's ignore the $x$-independent prefactor and just look at the function of $x$. Large wavelengths clearly correspond to large $x$ and small wavelengths to small $x$. But "large" and "small" relative to what? The claim is now that the dimensionless function $\overline{u}(x) = (e^{(1/x)}-1)^{-1}/x^5$ has no obvious "scale" for $x$ in it; therefore the most important values of $x$ (i.e. where the function is big) must be of order 1. This may seem like an outlandish claim, so let's check it. We can expand using a Taylor expansion in either $x$ or $1/x$ to find

$$\overline{u} = \begin{cases} e^{-(1/x)}/x^5 & x \ll 1 \\ \frac{1}{x^4} & x \gg 1 \end{cases} \tag{3}$$

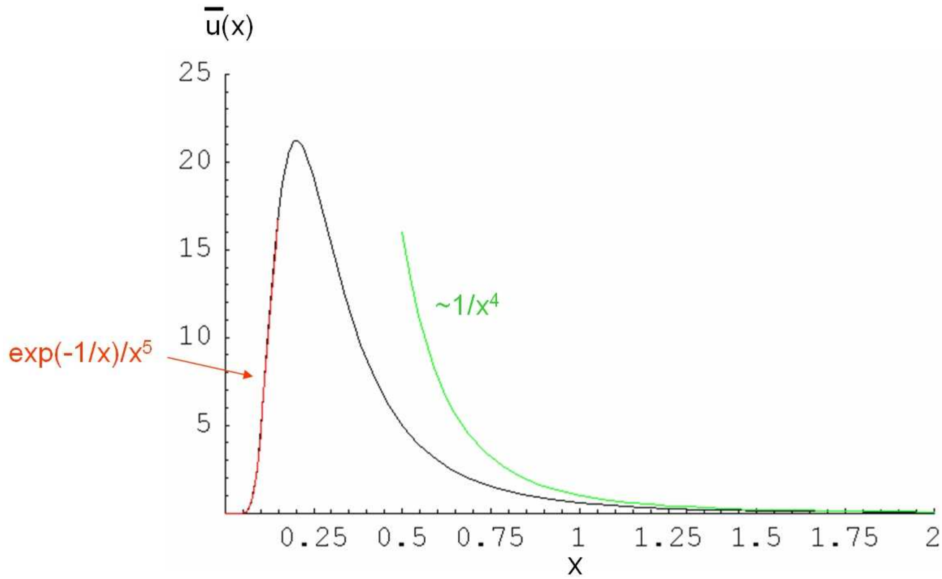


Figure 1: Energy density $\overline{u}$ of blackbody as function of reduced wavelength $x$, with asymptotic short- and long-wavelength approximations.

Note a couple of tricks we used to obtain these results. In the $x \ll 1$ limit, the exponential is huge and dominates the 1 in the denominator, which can be neglected. But the $\exp(-1/x)$ left over is divided by $x^5$. The numerator is going to zero, denominator also–which wins? Of course formally we should use l'Hospital's rule: what you will find is that the derivatives of the power on the bottom get bigger, but the derivatives on the top remain the same, hence the ratio indeed approaches zero as $x \to 0$. This is a special case of a general rule that when exponentials and power laws compete, "exponentials always win!" (just as powers always "win" over logs).

In the $x \gg 1$ limit, the exponential $\exp 1/x$ gets very close to 1, so we can expand in powers of $1/x$, $\exp 1/x \simeq 1 + 1/x + (1/x)^2/2 \ldots$. Hence the $\exp(1/x) - 1 \simeq 1/x$ and we get the result in Eq. 3.

A full plot of the function, as seen in the figure, reveals in fact that there is a peak around $x = 0.2$, which is indeed "of order 1", i.e. not of order $10^{-1}$ or $10^{1}$. As you see this criterion was pretty vague, because 0.2 is indeed closer to 0.1 than to 1, but a physicist would say, "ok, well, it might be $1/2\pi$, so if we take $2\pi$ equal

1 it works". This may sound ludicrous, but it is simply the statement that there is no other scale in the function. What do I mean by scale? Well, suppose I had a function

$$g(x) = \frac{1}{(x - 10,234)^2 + 1} \tag{4}$$

which cropped up in some problem. Then I would say, there is a scale, or dimensionless number 10,234 which is pretty important for the physics here for some reason. In our blackbody problem, however, it doesn't exist. The function is peaked near $x = 1$.

Does this mean a blackbody always radiates with a maximum intensity at a particular wavelength? Of course not. Depending on the temperature $T$, the $\lambda$ for which the maximum intensity is achieved can be quite different. To keep $x \sim 1$, if the temperature goes up the wavelength has to go down.

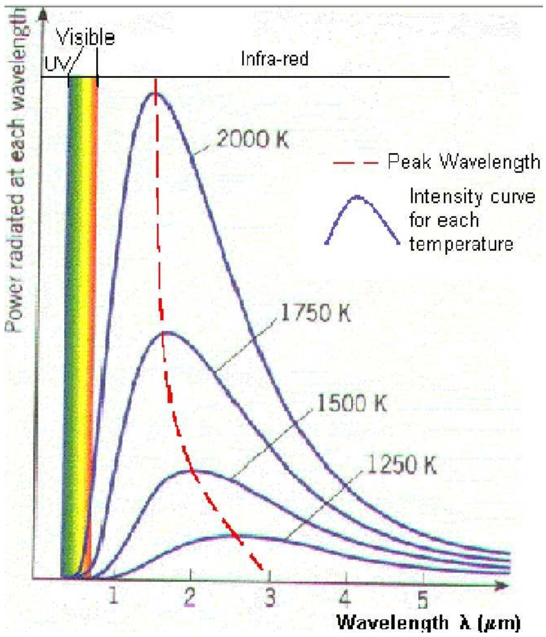


Figure 2: Energy density curve vs. wavelength in $\mu m$ for different temperatures in Kelvin.

## 2.2 Some tricks for doing integrals

### 2.2.1 Differentiation with respect to a parameter

Suppose a function of $x$ depends on a parameter $a$. Ex: $\sin ax$. If an integral converges, we can differentiate under the integral sign to obtain

$$F(a) = \int_{x_1}^{x_2} f(x,a)dx \tag{5}$$

$$\Rightarrow \frac{\partial F}{\partial a} = \int_{x_1}^{x_2} \frac{\partial f(x,a)}{\partial a} dx \tag{6}$$

If you know $F(a)$, you can derive results for the family of integrals $\int_{x_1}^{x_2} \partial^n f(x,a)/\partial a^n dx$. Sometimes you know an integral and would like to use it to do a related, harder integral; you can do so by introducing a parameter and differentiating with respect to it.

Ex: suppose you know or are given $\int_0^\infty (x^2+1)^{-1} dx = \pi/2$, but you want to know $\int_0^\infty (x^2+1)^{-2} dx$, which is not so obvious. Introduce

$$I(a) = \int_0^\infty \frac{dx}{x^2+a^2}, \tag{7}$$

then express in terms of $u = x/a$:

$$I(a) = \frac{1}{a}\int_0^\infty \frac{du}{u^2+1} = \frac{\pi}{2a}. \tag{8}$$

Now differentiate the original expression:

$$\frac{\partial I(a)}{\partial a} = -\int_0^\infty \frac{dx}{(x^2+a^2)^2}\cdot 2a \quad \text{but also} \quad \frac{\partial I(a)}{\partial a} = \frac{\partial(\pi/(2a))}{\partial a} = -\frac{\pi}{2a^2} \tag{9}$$

$$\Rightarrow \quad -\int_0^\infty \frac{dx}{(x^2+a^2)^2}\cdot 2a = -\frac{\pi}{2a^2} \tag{10}$$

$$\Rightarrow \quad \int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}. \tag{11}$$

Now note that the result holds for any value of $a$ we like, so we could put $a = 1$ to get

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}, \tag{12}$$

so in a pinch, e.g. if we were cast away on a desert island without tables of integrals or the internet, we could in principle derive this result, or corresponding ones for higher powers of the denominator, by taking further derivatives.

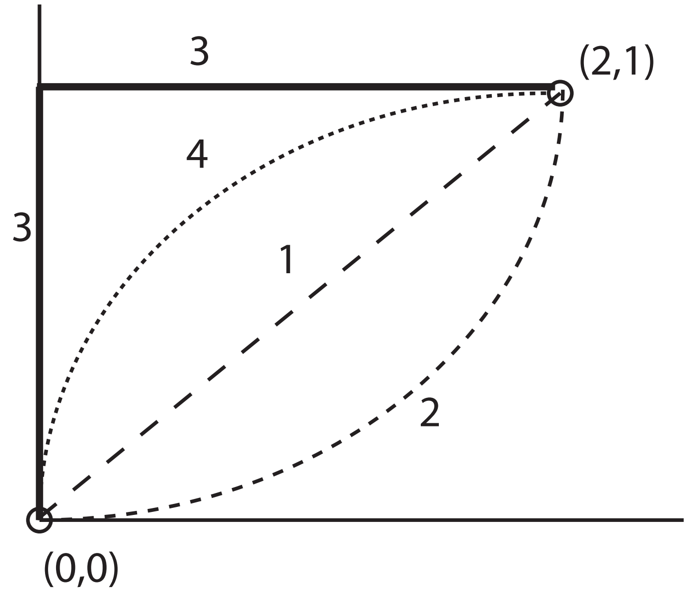


Figure 3: Paths from (0,0) to (2,1).

### 2.2.2 Integrals along a path

We consider the integral of a function of $x$ and $y$ along a path in 2-space (see Ch. 6, Sec. 8):

**Ex. 1:**

$$I = \int_{(0,0)}^{(2,1)} \left( xydx - y^2dy \right) \tag{13}$$

from $(0,0)$ to $(2,1)$ along different paths as shown. The paths are

1. 1 straight line
2. parabola $y = \frac{1}{4}x^2$
3. 2 straight lines along axes
4. $x = 2t^3$, $y = t^2$.

Q: Will the value of the line integral be the same along all paths?

A: No.

Path 1: $y = x/2 \Rightarrow\ dy = \frac{1}{2}dx$.

$$I_1 = \int_0^2 dx \left( x \cdot \frac{1}{2}x - \frac{1}{2}(x/2)^2 \right) = \int_0^2 dx \frac{3}{8}x^2 = 1 \tag{14}$$

Path 3: 3a) [(0,0) to (0,1)] $x = 0$, $dx = 0$, $I_{1a} = -\int_0^1 y^2 dy = -1/3$

3b) [(0,1) to (2,1)] $y = 1$, $dy = 0$ so $I_{3b} = \int_0^2 x\,dx = 2$.

So $I_3 = I_{3a} + I_{3b} = 5/3 \neq I_1$!

Paths: 2 & 4 – try yourselves before looking in book.

**Ex. 2:**

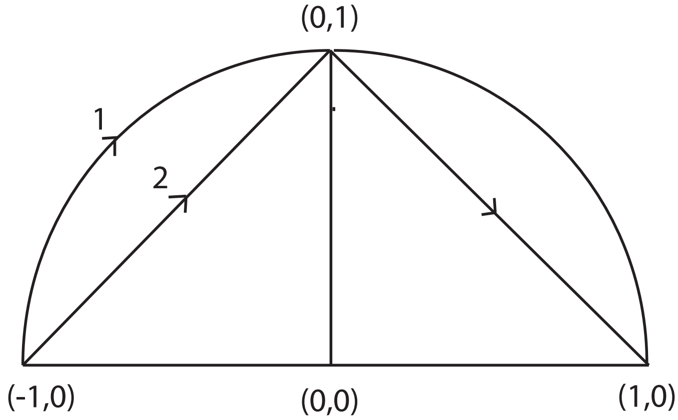


Figure 4: Paths from (-1,0) to (1,0).

Find the integral

$$J = \int \frac{x\,dy - y\,dx}{x^2 + y^2} \tag{15}$$

along the two paths shown.

1. Along semicircle: parameterize $x = \cos\theta$, $y = \sin\theta$. So

$$\frac{xdy - ydx}{x^2 + y^2} = d\theta \;\Rightarrow\; J = \int_\pi^0 d\theta = -\pi \tag{16}$$

2. Along straight lines. Check yourself, using $\int \frac{du}{u^2+1} = \tan^{-1} u$. Should find $J_1 = J_2$.

## 2.3 Fun with partial differentiation (see Boas ch. 4)

Notation: for $f = f(x, y)$,

$$\left(\frac{\partial f}{\partial x}\right)_y \tag{17}$$

is the partial derivative of $f$ with respect to $x$ with $y$ held fixed (constant).

Ex. 1: $f = x^2 - y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$

$$\left(\frac{\partial f}{\partial x}\right)_y = 2x \tag{18}$$

A little trickier: what is $(\partial f/\partial x)_\theta$ or $(\partial f/\partial \theta)_x$?

Method 1: express $f$ in terms of $x$ and $\theta$ first.

$$\begin{aligned} f &= x^2 - y^2 = x^2(1 - y^2/x^2) = x^2(1 - \tan^2\theta) && (19) \\ \Rightarrow \qquad \left(\frac{\partial f}{\partial x}\right)_\theta &= 2x(1 - \tan^2\theta) \neq \left(\frac{\partial f}{\partial x}\right)_y && (20) \end{aligned}$$

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Method 2: (works in principle even if method 1 fails because you can't solve for $f(x,\theta)$ in closed form): use chain rule:

$$\begin{aligned} f = f(x,y) \;\Rightarrow\; df &= \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy && (21) \\ \text{in this case} \qquad df &= 2x\,dx + 2y\,dy && (22) \end{aligned}$$

but if we consider $f = f(x,\theta)$ then

$$df = \left(\frac{\partial f}{\partial x}\right)_\theta dx + \left(\frac{\partial f}{\partial \theta}\right)_x d\theta \tag{23}$$

Now we need to compare (22) and (23). Now $y = x\tan\theta$, so $dy$ can be expressed in terms of $x$ and $\theta$ as follows:

$$\begin{aligned} dy &= \left(\frac{\partial y}{\partial x}\right)_\theta dx + \left(\frac{\partial y}{\partial \theta}\right)_x d\theta && (24) \\ &= \tan\theta\,dx + x\sec^2\theta\,d\theta. && (25) \end{aligned}$$

So putting these together we have

$$\begin{aligned} df &= 2x\,dx + 2y\,dy && (26) \\ &= 2x\,dx - 2y(\tan\theta\,dx + x\sec^2\theta\,d\theta) && (27) \\ &= \underbrace{2x(1-\tan^2\theta)}_{\left(\frac{\partial f}{\partial x}\right)_\theta}\,dx\,\underbrace{-2x^2\tan\theta\sec^2\theta}_{\left(\frac{\partial f}{\partial \theta}\right)_x}\,d\theta && (28) \\ & && (29) \end{aligned}$$

**Question** and frequent source of confusion: is $\frac{\partial\theta}{\partial x} = \left(\frac{\partial x}{\partial\theta}\right)^{-1}$? (In a situation where there is only one dependent variable this works, e.g take $x = \tan\theta$, $\theta = \tan^{-1} x$, it is easy to show that $d\theta/dx = (dx/d\theta)^{-1}$.) But in general,

**Answer**: no, *only true if the same variable is held constant in both cases.* For example,

$$x = r\cos\theta \Rightarrow \frac{\partial x}{\partial\theta} = -r\sin\theta = -y \text{ so this is } \left(\frac{\partial x}{\partial\theta}\right)_r \tag{30}$$

but also

$$\theta = \tan^{-1}\frac{y}{x} \Rightarrow \frac{\partial\theta}{\partial x} = \frac{-y/x^2}{1+y^2/x^2} = \frac{-y}{r^2} \text{ so this is } \left(\frac{\partial\theta}{\partial x}\right)_y \tag{31}$$

Since the same variable is not held constant, we do *not* find $\frac{\partial\theta}{\partial x} = \left(\frac{\partial x}{\partial\theta}\right)^{-1}$.

### 2.3.1 Chain rule, implicit differentiation

Ex. 2: Suppose $x + e^x = t$. What is $\frac{dx}{dt}$? $\frac{d^2x}{dt^2}$? If we could find $x(t)$ explicitly, no problem, but... we can't. Use differentials:

$$dx + e^x dx = dt \Rightarrow (1+e^x)dx = dt \Rightarrow \frac{dx}{dt} = \frac{1}{1+e^x} \tag{32}$$

To get second derivative, divide differential by $dt$ and differentiate wrt $t$:

$$\frac{dx}{dt} + e^x\frac{dx}{dt} = 1 \quad\Rightarrow\quad \frac{d^2x}{dt^2} + e^x\frac{d^2x}{dt^2} + e^x\left(\frac{dx}{dt}\right)^2 = 0 \tag{33}$$

$$\text{solve to find} \qquad \frac{d^2x}{dt^2} = -\frac{e^x}{(1+e^x)^3} \tag{34}$$

Ex 3:
If $w = f(ax+by)$, show that

$$b\left(\frac{\partial w}{\partial x}\right)_y = a\left(\frac{\partial w}{\partial y}\right)_x. \tag{35}$$

Let $u = ax + by$. We have $w = f(u)$, so

$$\left(\frac{\partial w}{\partial x}\right)_y = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = a\frac{\partial f}{\partial u} \tag{36}$$

$$\left(\frac{\partial w}{\partial y}\right)_x = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} = b\frac{\partial f}{\partial u}, \tag{37}$$

so Eq. (35) follows.

### 2.3.2 Thermodynamics

"Equation of state": $f(p,V,T)=0$, e.g. $pV-RT=0$ (ideal gas law) or $p-\frac{RT}{V-b}-\frac{a}{V^2}=0$ (van der Waals gas). Differential:

$$df=\left(\frac{\partial f}{\partial p}\right)_{V,T}dp+\left(\frac{\partial f}{\partial V}\right)_{p,T}dV+\left(\frac{\partial f}{\partial T}\right)_{p,V}dT=0 \tag{38}$$

Now do 3 different experiments:

1. hold $p$ fixed. $dp=0$ so

$$\left(\frac{\partial f}{\partial V}\right)_{p,T}dV_p = -\left(\frac{\partial f}{\partial T}\right)_{V,p}dT_p \tag{39}$$

$$\left(\frac{\partial V}{\partial T}\right)_p = -\frac{\left(\frac{\partial f}{\partial T}\right)_{p,V}}{\left(\frac{\partial f}{\partial V}\right)_{p,T}}=\frac{1}{\left(\frac{\partial T}{\partial V}\right)_p} \tag{40}$$

2. hold $V$ fixed. $dV=0$ so

$$\left(\frac{\partial T}{\partial p}\right)_V = -\frac{\left(\frac{\partial f}{\partial p}\right)_{T,V}}{\left(\frac{\partial f}{\partial T}\right)_{p,V}}=\frac{1}{\left(\frac{\partial p}{\partial T}\right)_V} \tag{41}$$

3. hold $T$ fixed. $dT=0$ so

$$\left(\frac{\partial p}{\partial V}\right)_T = -\frac{\left(\frac{\partial f}{\partial V}\right)_{p,T}}{\left(\frac{\partial f}{\partial p}\right)_{T,V}}=\frac{1}{\left(\frac{\partial V}{\partial p}\right)_T}. \tag{42}$$

Now examine product of two such derivatives, using above results

$$\left(\frac{\partial p}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_p = \left[-\frac{\left(\frac{\partial f}{\partial V}\right)_{p,T}}{\left(\frac{\partial f}{\partial p}\right)_{T,V}}\right]\left[-\frac{\left(\frac{\partial f}{\partial T}\right)_{p,V}}{\left(\frac{\partial f}{\partial V}\right)_{p,T}}\right]=\frac{\left(\frac{\partial f}{\partial T}\right)_{p,V}}{\left(\frac{\partial f}{\partial p}\right)_{T,V}} \tag{43}$$

$$= -\left(\frac{\partial p}{\partial T}\right)_V. \tag{44}$$

So independent of any particular equation of state $f$ we always have relation among partial derivatives

$$\left(\frac{\partial p}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_p\left(\frac{\partial T}{\partial p}\right)_V=-1 \tag{45}$$