5 Vector and scalar fields

5.1 scalar fields

A "scalar field" is a fancy name for a function of space, i.e. it associates a real number with every position in some space, e.g. in 3D $\phi = \phi(x, y, z)$. We've already encountered examples without calling them scalar fields, e.g. the temperature $T(x, y)$ in a metal plate, or the electrostatic potential $\phi = \phi(x, y, z)$. The gravitational potential is another, and it's frequently convenient to think about potential "landscapes", imagining that a set of hills is a kind of paradigm for a varying potential, since the height in this case scales with the potential $mgh(x, y)$ itself.

Formally, scalar is a word used to distinguish the field from a vector field. We can do this because a scalar field is *invariant* under the rotation of the coordinate system:

$$\phi'(x', y', z') = \phi(x, y, z). \tag{1}$$

In other words, I may label the point on top of one of the hills by a different set of coordinates, but this doesn't change the height I assign to it. This is in contrast to a vector field, where the values of the components do change in the new coordinate system, as we have discussed.

5.1.1 gradients of scalar fields

If you're standing on the hill somewhere, say not on the top, there's one direction in xy space which gives you the direction of the fastest way down. This vector is $\vec{\nabla}\phi$, where ϕ is the gravitational potential. Consider the differential $d\phi$ in 2D:

$$
\begin{aligned}
d\phi &= \left(\frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial \phi}{\partial y}\right) dy = \underbrace{\left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y}\right)}_{\vec{\nabla}\phi} \cdot \underbrace{\left(\hat{i}dx + \hat{j}dy\right)}_{d\vec{r}} \equiv \vec{\nabla}\phi \cdot d\vec{r} \\
&= |\vec{\nabla}\phi||d\vec{r}| \cos\theta.
\end{aligned}
\tag{2}
$$

where θ is the angle between the gradient $\vec{\nabla}\phi$ and the change in position $d\vec{r}$, so we see that the general change of ϕ is the *projection* of the gradient onto the direction of whatever change one is making; this is sometimes called a *directional derivative.*

One important way to remember about gradients of scalar fields is that *they are always perpendicular to lines of constant scalar field.* You know this if you've
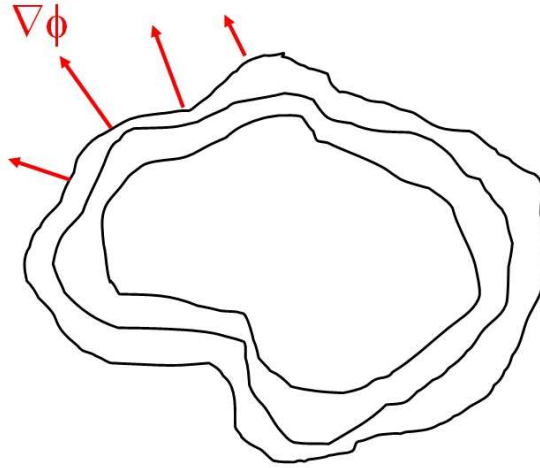
Figure 1: Contours of constant ϕ and gradient $\vec{\nabla}\phi$.

ever used a topographical map to navigate in the woods. Figure 1 shows such a set of contours (lines of equal height, or gravitational potential) which are sort of concentric. The arrows shown are the gradients of the height or potential at the points shown. Note a couple of things:

- the arrows point out, so the "map" must be of a valley in the center, since the gradient points in the direction of steepest ascent.

- $\nabla\phi$ points perpendicular to the lines of constant ϕ.

- The arrows are longer ($\nabla\phi$ is bigger) where the rise is steeper, i.e. the contours are closer together.

Let's do some examples:

Ex. 1: $\phi = r = \sqrt{x^2 + y^2 + z^2}$, so

$$\vec{\nabla}r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z} = \frac{1}{r}\left(\hat{i}x + \hat{j}y + \hat{k}z\right) = \frac{\vec{r}}{r} = \hat{r}, \tag{3}$$

so r increases fastest along $\hat{r}$ — no surprise.

Ex. 2: Fermi velocity.

In a metal, the electrons make up a kind of a gas, almost free. Even at $T = 0$ they are moving, because the Pauli principle prevents them from being in the lowest (zero) momentum state. The figure shows the allowed quantum mechanical energy states of an electron in a metal. There can only be one spin up and one spin down

in each momentum state, somewhat like an atom. For a given hunk of material, which has a given number of electrons in it, the highest such occupied level has an energy referred to as the Fermi energy, ϵ_F. In a typical light metal, the Fermi energy is quite large, of order a few eV. In general this defines some surface in momentum space, because the energy-momentum relation in the metal is $\epsilon(\vec{p})$. For concreteness, let's assume

$$\epsilon(\vec{p}) = \frac{p_x^2 + p_y^2}{2m_\perp} + \frac{p_z^2}{2m_z},\tag{4}$$

where "$m_\perp$" and "m_z" don't really represent the components of a vector mass (mass is a scalar, right?), but some effective coefficients coming from solving the Schrödinger equation of quantum mechanics properly, which happen to have dimensions of mass.
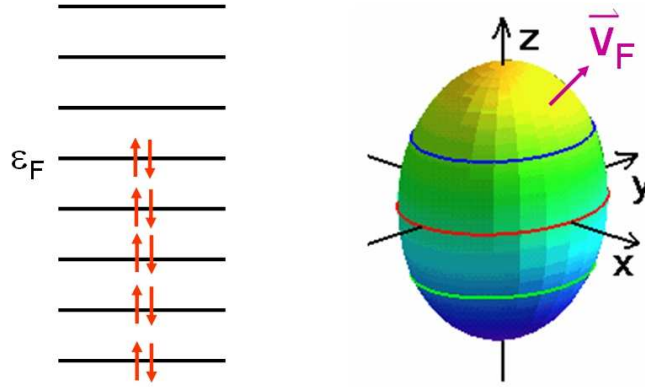

Figure 2: Left: states allowed by Pauli principle in a metal, and definition of Fermi energy ϵ_F as highest occupied state. Right: ellipsoid representing surface in momentum space where $\epsilon(\vec{p}) = \epsilon_F$.

The Fermi velocity is now defined as $\vec{\nabla}\epsilon(\vec{p})|_{\epsilon=\epsilon_F}$. If $m_\perp = m_z$, it would point radially in the $\hat{p}$ direction, but for the ellipsoidal case as shown, we can calculate it to be

$$\vec{\nabla}\epsilon(\vec{p})|_{\epsilon=\epsilon_F} = \hat{i}\frac{p_x}{m_\perp} + \hat{j}\frac{p_y}{m_\perp} + \hat{k}\frac{p_z}{m_z}|_{\epsilon=\epsilon_F},\tag{5}$$

where the p_x, p_y, and p_z are the values of these quantities on the ellipsoid $\epsilon(p_x, p_y, p_z) = \epsilon_F$. The Fermi velocity is always perpendicular to the Fermi surface. In a simple metal, this typical velocity of a conduction electron has a magnitude of about $1/100$ to $1/10$ of the speed of light.

5.1.2 Transformation of scalar fields under rotations.

How does a scalar field transform when the coordinate system is rotated? Unlike the components of a vector field, see last week's notes, a scalar field transforms as

$$\phi'(x', y', z'...) = \phi(x, y, z) \tag{6}$$

, i.e. it is invariant! Consider what this really means: suppose you have a map, and are looking at the height of a particular hill, which on your map has coordinates x and y. If someone else has a map with a rotated coordinate system, coordinates x' and y', the height of that particular hill doesn't change, just the coordinates!