8 Electrodynamics

Read: Boas Ch. 6, particularly sec. 10 and 11.

8.1 Maxwell equations

Some of you may have seen Maxwell's equations on t-shirts or encountered them briefly in electromagnetism courses. These equations were written down for the first time by Scottish physicist James Clerk Maxwell in his "Treatise on Electricity and Magnetism" (1873), and they caused a stir because his new equations proved that light was an electromagnetic phenomenon. Imagine that you had no clue that light and all the phenomena of electricity and magnetism you knew from the laboratory were related, and someone showed you that you could calculate the speed of some weird electromagnetic wave solutions to these differential equations, and show that the speed was exactly that of light (which had been measured astronomically). Spectacular! With the new tools you possess you can understand the equations at a deeper level.

Suppose charge is increasing at some rate within a given volume τ. Assume that we have no "sources" or "sinks" of charge in the system. This means that it has to come from outside the region. The amount created inside per time has to show up as a flux of the charge through the boundary of the volume coming from outside (make sure you understand the sign):

$$\frac{\partial}{\partial t}\int_{\text{vol } \tau} \rho(\vec{r},t)d\tau = -\int_{\text{surf. } \partial\tau} \vec{j}\cdot d\vec{a} = -\int_{\tau} \vec{\nabla}\cdot\vec{j}\, d\tau \tag{1}$$

where the last equality follows from the divergence theorem. Now since we did this for an arbitrary volume τ, it must hold locally:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0, \tag{2}$$

the so-called "*equation of continuity*". This is not thought of as one of Maxwell's equations, because it doesn't contain the electromagnetic fields $\vec{E}$ and $\vec{B}$, but merely expresses the conservation of charge. Here are the standard 4:

1. Gauss's law. For pt. charge,

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r} \Rightarrow \int_{\text{closed}} \vec{E}\cdot d\vec{a} = \frac{q}{4\pi\epsilon_0}\int \underbrace{\frac{\hat{r}\cdot d\vec{a}}{r^2}}_{d\Omega} = \frac{q}{\epsilon_0}$$

Divergence theorem then says

$$\int_\tau (\vec{\nabla} \cdot \vec{E})\, d\tau \;=\; \frac{1}{\epsilon_0} \underbrace{\int_\tau \rho(\vec{r}) d\tau}_{q} \Rightarrow \underbrace{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}}_{\text{Maxwell I}} \tag{3}$$

There are two mathematical subtleties I swept under the rug in this "proof". First I did it for a point charge, but expressed things in terms of a general charge density at the end. You can go back and convince yourself that if you say the $\vec{E}$-field is a sum of many small charge elements dq each producing a field falling off like $1/r^2$ from itself, you get the same answer. Secondly, in the last step we jumped from a statement about equality of integrated quantities to a statement about the equality of the integrands. Normally this is a no-no, but here it's ok since we are talking about a relation which is valid for any ρ distribution. We'll see the underlying reason for this kind of argument when we talk about function spaces.

2. No magnetic charge.

$$\underbrace{\vec{\nabla} \cdot B = 0}_{\text{Maxwell II}}. \tag{4}$$

You can make an analogous mathematical argument for the magnetic field, but since we don't know about any particles which carry magnetic charge (yet), we set the right hand side equal to zero.

3. Faraday's law.

$$\oint_{\text{loop}} \vec{E} \cdot d\vec{r} \;=\; -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \int_{\text{surface A}=\partial \text{ loop}} \vec{B} \cdot d\vec{a}. \tag{5}$$

Now use Stokes' theorem:

$$\int_A (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = \oint \vec{E} \cdot d\vec{r} = -\int_A \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}, \tag{6}$$

where we assumed that the loop over which the $\oint$ was taken was fixed in time (only field was changing), so we could bring the time derivative inside the integral and apply it to the $\vec{B}$ only. But now again we have integrals left and right over the same surface (this time), and the surface A is arbitrary, so the only way the equation can hold is if

$$\underbrace{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}_{\text{Maxwell III}} \tag{7}$$

4. <u>Ampère's Law.</u>

$$\oint_{\text{loop}} \vec{B} \cdot d\vec{r} = \mu_0 I_{\text{encl.}} = \mu_0 \int_{A \text{ enclosed by loop}} \vec{j} \cdot d\vec{a}. \tag{8}$$

Using our previous reasoning and applying Stokes' theorem, we might come to the conclusion that

$$\int_A \vec{\nabla} \times \vec{B}\, d\vec{a} = \mu_0 \int_A \vec{j} \cdot d\vec{a} \quad \Rightarrow \quad ??? \;\; \vec{\nabla} \times \vec{B} = \mu_0 j \;\; ??? \tag{9}$$

You might worry about this conclusion, because if we take the divergence of both sides, we see that $\vec{\nabla} \cdot j = 0$ always, whereas the continuity equation says there has to be another term $d\rho/dt$ when the charge density is changing locally in time. What happened to it? Well, turns out the version of Ampère's law we started with wasn't the most general one; it's valid only for stationary currents and fields. We need to add something else if things are changing with time. To guess what to add, consider the AC circuit shown,

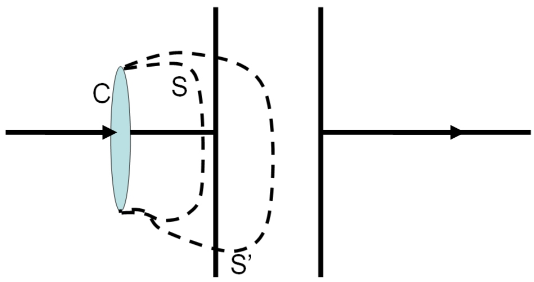

Figure 1: Geometry for calculation of displacement current. C is a loop containing a wire leading to a capacitor, S is an open surface enclosing C, as is S'. Only S' encloses the capacitor, however.

The current flowing through the wire is the time derivative of the charge Q on the capacitor, $I = dQ/dt$. If we apply Stokes' theorem to S and use the stationary form of Ampère's law we get:

$$\oint_C \vec{B} \cdot d\vec{r} = \int_S \vec{\nabla} \times \vec{B} \cdot d\vec{a} = \mu_0 \int_S \vec{j} \cdot d\vec{a} = \mu_0 I, \tag{10}$$

no problem, but if we were to do the same thing with S' we would get zero, since there is no current actually flowing through S'!!! Remember we can't argue with Stokes' law–that's mathematics. The "paradox" suggests that there is a missing term in Ampère's law which "turns on" when there is a time changing electric field, such as that which exists on the capacitor plate when charge is building up there. Maxwell guessed a generalization of Ampère's law:

$$\vec{\nabla} \times \vec{B} = \mu_0(\vec{j} + \vec{j}_d) \quad ; \qquad \vec{j}_d \equiv \epsilon_0 \frac{\partial \vec{E}}{\partial t}. \tag{11}$$

Now applying Stokes' law we get that the right hand side should be just like 10, except we replace $\vec{j}$ with $\vec{j} + \vec{j}_d$. The second term $\vec{j}_d$ is called the displacement current; it has the dimensions of a current, but does not correspond to transport of charge. And we find that it doesn't depend any more which surface we choose, since

$$\int_S (\vec{j} + \vec{j}_d) \cdot d\vec{a} = \int_{S'} (\vec{j} + \vec{j}_d) \cdot d\vec{a}. \tag{12}$$

For the first term $\frac{\partial \vec{E}}{\partial t} = 0$ over the surface S, which may be taken far from the capacitor, the displacement current is zero, but the physical current is nonzero. For S', which passes mostly near the capacitor's surface, there's no physical current, but the charge is building up so $\vec{E}$ is changing with time. Thus the current is exclusively displacement in nature.

Have we fixed the continuity problem? Take the divergence of both sides of (11), and using $\vec{\nabla} \cdot \vec{\nabla} \times \vec{v} = 0 \quad \forall \vec{v}$, we find

$$\begin{aligned} 0 &= \mu_0 \vec{\nabla} \cdot \vec{j} + \vec{\nabla} \cdot \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{E} && (13) \\ &= \mu_0 \left(\vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} \right), && (14) \end{aligned}$$

where in the last step I used Coulomb's law $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$. So continuity is satisfied (charge conserved), so we can all rest easy in our beds. For completeness, let me then record our answer from Maxwell for the modified Ampère's law:

$$\underbrace{\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}}_{\text{Maxwell IV}}. \tag{15}$$

Some examples of the math of E& M:

- Gauss law. Ball of radius R, constant chg. density ρ, could sum up the electric field contributions $d\vec{E}$ from all infinitesimal charge elements dq, or use divergence theorem and Poisson equation $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$. For a "Gaussian sphere" A, radius $r > R$,

$$\int_A \vec{E} \cdot d\vec{a} \;=\; \int_A E(r)da = 4\pi r^2 E(r) = \int_\tau \rho d\tau = \frac{4}{3}\pi R^3 \frac{\rho}{\epsilon_0} \tag{16}$$

$$\therefore E(r) = \frac{\frac{4}{3}\pi R^3 \rho}{4\pi\epsilon_0 r^2} = \frac{Q}{4\pi\epsilon_0 r^2}, \tag{17}$$

where recall the symmetry argument that the field must be radial due to the spherical symmetry of the charge distribution was crucial. For $r < R$,

$$4\pi r^2 E = \frac{4}{3}\pi r^3 \frac{\rho}{\epsilon_0} \quad \Rightarrow \quad E = \frac{\rho}{3\epsilon_0} r \tag{18}$$

Reminder: how do we find the potential ϕ, given $\vec{E}$?

$$\phi_B - \phi_A = -\int_A^B \vec{E} \cdot d\vec{r} \tag{19}$$

Choose reference point $\phi(r \to \infty = 0$,

$$\phi(r > R) \;=\; -\int_\infty^r \frac{Q}{4\pi r^2 \epsilon_0} dr = \frac{Q}{4\pi\epsilon_0 r} \tag{20}$$

$$\phi(r < R) - \phi(R) \;=\; -\int_R^r \frac{\rho}{3\epsilon_0} r dr = -\frac{\rho}{6\epsilon_0}(r^2 - R^2) \tag{21}$$

$$= -\frac{Q}{6\epsilon_0}\left(\frac{r^2 - R^2}{\frac{4}{3}\pi R^3}\right) \tag{22}$$

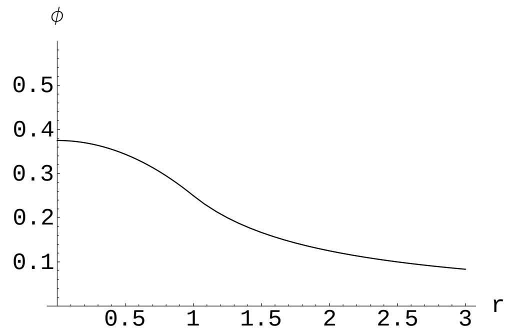

Figure 2: Potential of solid sphere of charge in units of $Q/\pi\epsilon_0 R$.

- EM waves. Maxwell's great achievement! Consider free space, with $\rho = 0$ $\vec{j} = 0$, such that $\vec{\nabla} \cdot E = 0$, $\vec{\nabla} \cdot \vec{B} = 0$. Consider Ampère's law and Poisson eqn. for B-field ("no magnetic monopoles" law):

$$\vec{\nabla} \cdot \vec{B} = 0 \quad ; \quad \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{23}$$

$$\underbrace{\vec{\nabla} \times (\vec{\nabla} \times \vec{B})}_{\vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B}} = \mu_0 \epsilon_0 \vec{\nabla} \times \frac{\partial \vec{E}}{\partial t} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \underbrace{(\vec{\nabla} \times \vec{E})}_{-\frac{\partial \vec{B}}{\partial t}} = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{B}, \tag{24}$$

where I used a common vector identity, setting $\vec{\nabla} \cdot \vec{B} = 0$, and the last step follows from Faraday's law. This is now a differential equation for the components of $\vec{B}$ only,

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{B}. \tag{25}$$

Compare with the wave equation we discussed earlier, for propagating waves in 1D with speed c,

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c}^2 \frac{\partial^2 u}{\partial t^2} = 0, \tag{26}$$

and you will see that each component of $\vec{B}$ obeys its own wave equation, meaning that propagating magnetic waves are a property of Maxwell's equations. By comparing with (26), we see they have a speed

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 2.997925 \times 10^8 \text{ m/s}. \tag{27}$$

The quantities μ_0 and ϵ_0 are measured in the laboratory, and 2.997925×10^8 m/s is very close to the known speed of light (from astronomical observations) in Maxwell's time. It's vital to understand that the conclusion that light waves are electromagnetic waves, so mundane sounding to us today, was a dramatic discovery in Maxwell's day. It unified the understanding of disparate phenomena (light, electricity & magnetism) which had previously been thought independent. Note also that the speed in the equations is not given relative to some medium, a fact which had profound consequences for Einstein's thinking. Finally, note that you will derive an identical wave equation for the electric field $\vec{E}$ on the homework. A propagating EM wave in free space has the same magnitude of E and B, with their polarizations perpendicular.

- Gauge transformations.

$$
\begin{aligned}
\text{Maxwell II: } \vec{\nabla}\cdot\vec{B} &= 0 \quad \Rightarrow \quad \vec{B}=\vec{\nabla}\times\vec{A}, \quad \vec{A}= \text{ vector pot.} && (28)\\
\text{Maxwell III: } \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \quad \Rightarrow \quad \vec{E}=-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t} && (29)\\
& \qquad\qquad\qquad\quad \phi = \text{ scalar potential.} && (30)
\end{aligned}
$$

Note: $\vec{A},\phi$ are not unique. If Λ is any function of space and time, we can make the changes

$$
\vec{A}\rightarrow\vec{A}+\vec{\nabla}\Lambda, \quad \phi\rightarrow\phi-\frac{\partial\Lambda}{\partial t} \tag{31}
$$

without changing the fields $\vec{E}$, $\vec{B}$. Check!