Variance of the cross-correlation in presence of correlated noise

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1 Statement of the problem

In case of two GW detectors, let say *L* and *H*, the output of each detector is a time series, which is a mixture of the SGW signal (*h*) and noise (*n*)

$$x_{L,H}(t) = h_{L,H}(t) + n_{L,H}(t)$$

Assuming no correlation between signal and noise, the cross-correlation *S* has a mean value

$$\langle Y \rangle = \langle h_L h_H \rangle + \langle n_L n_H \rangle$$

and variance

$$V = \langle Y^2 \rangle - \langle Y \rangle^2 .$$

One can see, the noise term $\langle n_L n_H \rangle$ may bias the mean value of the cross-correlation. This is one problem. We consider a different problem. Our goal is to calculate the variance of *Y*, which could be affected by correlated noise as well as the mean value.

2 Y variance

The cross-correlation between two detectors is

$$Y = \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' x_L(t') x_H(t) Q(t - t', \Omega_L, \Omega_H),$$

where T is the integration time and Ω_H (Ω_L) is the orientation of H (L) interferometer. The integration kernel *Q* is selected to maximize the correlation due to the stochastic GW signal. In Fourier domain it reads

$$Q(f, \Omega_L, \Omega_H) = \lambda \frac{|f|^{-3} \Omega_{GW}(f) \gamma(f, \Omega_L, \Omega_H)}{P_L(f) P_H(f)},$$

where $\Omega_{GW}(f)/f$ is proportional to the GW differential energy density, γ is the detector overlap reduction function, P_L, P_H are the spectral densities of the detector noise and λ is the normalization constant. Then the full expression for the cross-correlation is

$$Y = \int_{-\infty}^{\infty} \tilde{x}_L(f) \tilde{x}_H^*(f) Q(f, \Omega_H, \Omega_L) df .$$

Lets introduce the "filtered" data:

$$\tilde{X}_{L,H}(f),$$

so the cross-correlation at zero lag time is

$$Y = \int_{-\infty}^{\infty} \tilde{X}_L(f) \tilde{X}_H^*(f) df = \int_{-T/2}^{T/2} X_L(t) X_H(t) dt .$$

Lets call "the correlation statistics *S(t)*" a sample-by-sample product of the filtered data

$X_L(t)$ and $X_L(t)$

$$S(t) = X_L(t) X_H(t)$$

We can introduce the autocorrelation function of the statistics $S(t)$

$$A(\tau) = \int_{-\infty}^{\infty} \left|\widetilde{S}(f)\right|^2 \exp(-2\pi j f \tau) df \ .$$

It can be normalized to unity at τ=*0*

$$a(\tau) = A(\tau)/A(0),$$

where $A(0)$, in fact, is the theoretical variance calculated by the stochastic DSO

$$A(0) = \sigma_0^2 = \int_{-\infty}^{\infty} \left|S(f)\right|^2 df = \int P_H P_L Q^2(f, \Omega_L, \Omega_H) df \ .$$

Defining the *correlation time* $\boldsymbol{T_c}$ as

$$T_c = \int_0^{\infty} a(\tau) d\tau ,$$

the cross-correlation variance can be estimated as

$$\mathrm{var}(Y) \approx \sigma_0^2 \frac{2T_c}{\Delta t} ,$$

where Δt is the data sampling interval.

If no correlation processes are present in the data, the auto-correlation function satisfies

$$a(0) = 1 , \quad a(\tau > 0) = 0$$

and $\mathrm{var}(Y) = \sigma_0^2$. When some correlated process is present with correlation time scale T_s, much less then the observation time, the correlation time

$$T_c(\tau_s) = \int_0^{\tau_s} a(\tau) d\tau$$

increases as the function of the integration limit and then saturates when $\tau_s > T_S$. So this sort of correlated noise does not affect the mean value of Y but it may affect its variance. The relative variance increase due to correlated noise, compare to the un-correlated noise, is

$$\nu(\tau_s) = \frac{2}{\Delta t} \int_0^{\tau_s} a(\tau) d\tau = \frac{2T_c}{\Delta t} .$$

Note, factor ν may be less then 1 if signal from a correlated process, like large injected stochastic background signal, is large compared to the uncorrelated noise. Also the value of ν, too different from the unity, may require the usage of a different optimal filter.

Measuring the autocorrelation function *a(τ)* we can estimate the contribution from the correlated noise and estimate the *Y* variance as

$$\mathrm{var}(Y) = \nu(\tau_s) \cdot \sigma_0^2 .$$

The justification of this approach in case of the non-parametric sign cross-correlation is described in section 3.3 of "A cross-correlation technique in wavelet domain…." by S.Klimenko et al, gr-qc /0208007 or
http://www.phys.ufl.edu/LIGO/stochastic/sign05.pdf