

Contour Dynamics of Two-Dimensional Incompressible Systems

Christina Skowronski ^(a,b), Alan Dorsey ^(b), and Carlos Wexler ^(b)

(a) Rutgers University, New Brunswick, NJ

(b) University of Florida, Gainesville, FL

Studying models of incompressible systems is very important as there are many systems in different areas of physics that may be regarded as incompressible. Two particular examples are vortex patches in ideal fluids [1] and two-dimensional electrons systems in the presence of a magnetic field [2]. Since these systems are incompressible, they are easier to study because their density is “piecewise constant”. This means that the density of the fluid is observed to remain constant. Thus, the system will have a constant volume and mass. In many common fluids compression effects may be neglected and therefore, an incompressible fluid model may be used to study the system. Mathematically, incompressibility means that the divergence of the velocity of the fluid is equal to zero [3, 4].

The advantage of studying two-dimensional incompressible systems is that the modes that exist in the bulk of the fluid are very high energy modes and can be neglected, as it is the lower energy modes that are mainly of concern. This allows the system to be characterized by the motion of its boundary alone. Thus, the two-dimensional problem can be reduced to a one dimensional problem, which is easier to study both analytically and through numeric methods.

In some incompressible fluids there are vortex patches that may arise. These two dimensional vortex patches have a finite area that remains constant. In time, the patch may change shape, however the area does not change. Thus, it is possible to study how these vortex patches evolve in time by studying the boundaries alone. One can assume that there will be a constant vorticity ω_p inside the boundary while outside the boundary, the vorticity is zero. Then the following equations are obtained:

$$\nabla \times \mathbf{v} = \omega_p, \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0 \implies \mathbf{v} = \nabla \times \mathbf{A}. \tag{2}$$

Outside the boundary $\nabla \times \mathbf{v} = 0$. From these starting equations, it is possible to derive an expression for the velocity of a point on the boundary. This is

done by substituting equation (2) into equation (1) and solving for \mathbf{A} . After finding \mathbf{A} , the curl is taken, and an expression for the velocity is found. When the velocity of each point on the boundary is known, the position of each point on the boundary can easily be found from

$$\mathbf{v}(\mathbf{r}) = \frac{\omega_p}{2\pi} \oint_{\Gamma} ds' \hat{\mathbf{t}}(s') \ln \left| \frac{\mathbf{r}(s) - \mathbf{r}(s')}{r_0} \right|. \quad (3)$$

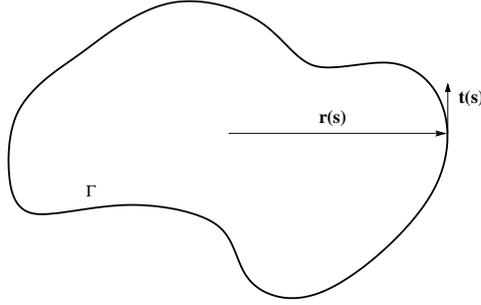


Figure 1: Shows the tangent vector $\hat{\mathbf{t}}(s)$ and position vector $\hat{\mathbf{t}}$ for an arbitrary contour.

From equation (3), one notices that the velocity of a point on the boundary is dependant on the position of the boundary and the tangent vector. Though this equation may look simple, it can become difficult or even impossible to solve analytically depending on the shape of the boundary. Thus, in order to solve this integral equation more easily, one can use a numerical method.

There are two basic steps in solving equation (3) numerically, the discretization of space and the discretization of time. First, the spatial discretization is done in order to find the velocity of the boundary. When the velocity at each point is known, time is stepped forward in order to find the position of the boundary at some later time.

As a first step in the spatial discretization, the contour is divided up into n points. It is possible to express the tangent vector at a point s_j on the boundary in terms of the position of its two neighboring points s_{j-1} and s_{j+1} , and the distance Δ between two points, thus the tangent vector is

$$\hat{\mathbf{t}}(s_j) = \frac{\mathbf{r}(s_{j+1}) - \mathbf{r}(s_{j-1})}{2\Delta}. \quad (4)$$

To complete the spatial discretization, the integral can be represented as a sum over the j points on the boundary:

$$\mathbf{v}(\mathbf{r}(s_i)) = \frac{\omega_p}{4\pi} \sum_j [\mathbf{r}(s_{j+1}) - \mathbf{r}(s_{j-1})] \ln \left| \frac{\mathbf{r}(s_i) - \mathbf{r}(s_j)}{r_0} \right|. \quad (5)$$

However, from equation (5) one may notice that there is a singularity that arises when $j = i$. It is important to look out for such singularities when integrating numerically. In order to compute the integral, it will be necessary to calculate the integral analytically in the neighborhood of the singular point, while outside this region the integral is done numerically. The region around the singular point can be approximated by some analytic function such as a line, a circle, or a polynomial, depending on the shape of the contour. Also, equation (5) shows the simplest way of computing an integral numerically. Depending on the boundary, a more accurate method of numeric integration, such as Simpson's Rule, may be necessary.

Following the spatial discretization, the next step is the time discretization which will give the position of the boundary at a later time in increments of Δt . Equations (6) and (7) show how this may be done basically from the definition of the derivative,

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}(\mathbf{r}(s_i)), \quad (6)$$

$$\mathbf{r}(s_i, t + \Delta t) = \mathbf{r}(s_i, t) + \Delta t \frac{d\mathbf{r}}{dt}. \quad (7)$$

Using the results of the spatial and time discretizations, one may write a computer program to do the calculations given initial conditions.

The simplest boundary to consider is a circle. The advantage of first studying the circle is that it may be relatively easily solved analytically. Thus, it is helpful in determining if the program that was written is functioning correctly. In order to calculate the integral, I , in the region near the singularity, the following expression is derived:

$$I = 2 \left[\Delta \ln \left(\frac{\Delta}{r_0} \right) - \Delta \right], \quad (8)$$

where Δ is a small distance on the contour from the singular point.

Using the initial condition $\omega_p = 1$, and where the boundary is a circle of unit radius, refer to figure 2, analytically the result for the velocity at each point on the boundary is given by a circle of radius 0.5. When the calculation is done numerically, the result is approximately equivalent for a large number of points n . Figure 3 shows both the exactly calculated velocity and the numeric approximation with $n = 2000$.

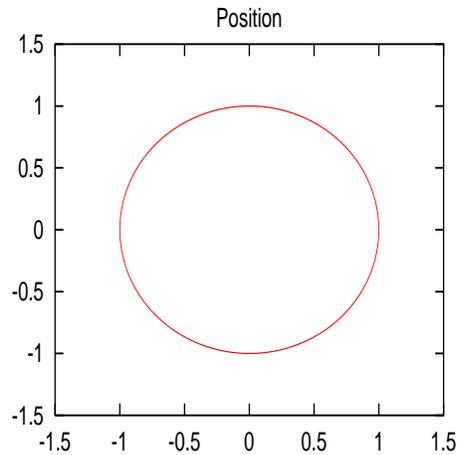


Figure 2: The initial position of a circular boundary of radius 1.

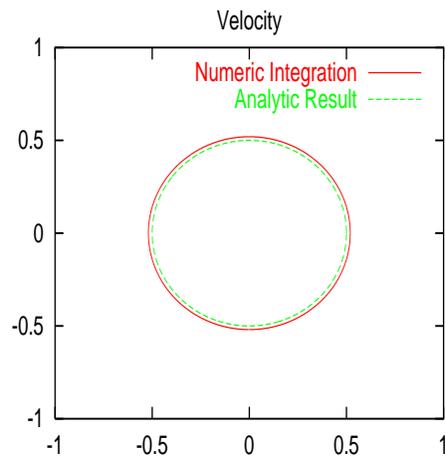


Figure 3: The velocity of each point on the boundary, both the numeric approximation and exact results are shown.

Figure 4 shows how the accuracy is dependant on the number of points taken on the boundary. One can see that as the number of points is increased, the magnitude of the velocity converges to the expected value of 0.5. By using a more accurate numeric integration technique, the velocity magnitude would converge more quickly. In addition, the circle boundary does not deform in time, and remains a circle. More accurately, the boundary will rotate, however, a plotted rotating circle will look the same.

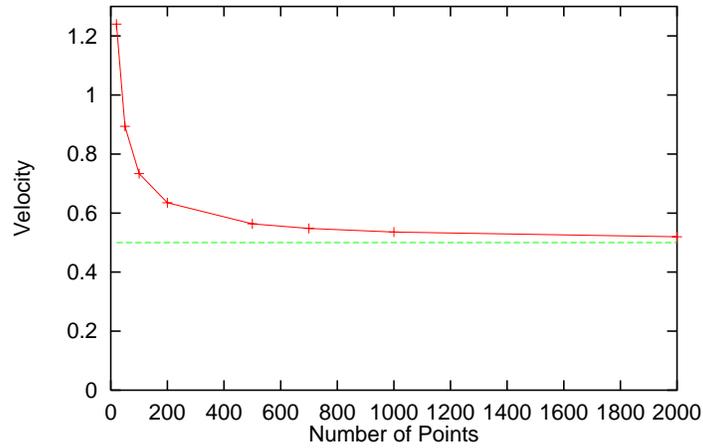


Figure 4: Shows how the accuracy of the numeric approximation is dependant upon the number of points taken.

The second important boundary shape to look at is the ellipse. Like the circle, the shape of the ellipse should be preserved as time evolves. Figure 5 shows the position of three ellipses of different eccentricities, and figure 6 shows their velocities as they were calculated by a numeric method. The velocities of the ellipses seem to be very circular.

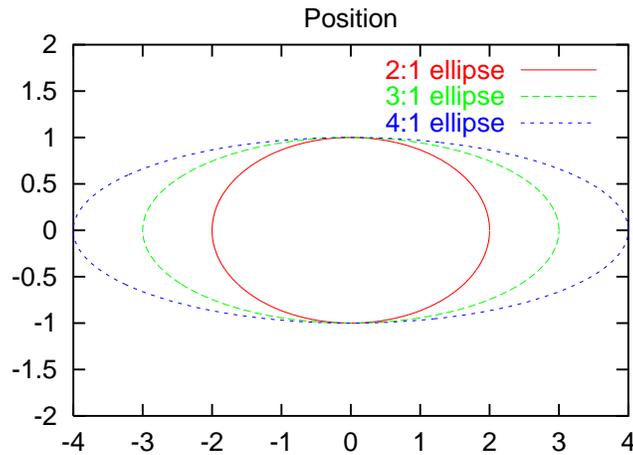


Figure 5: Initial position of elliptical boundaries of varying eccentricity.

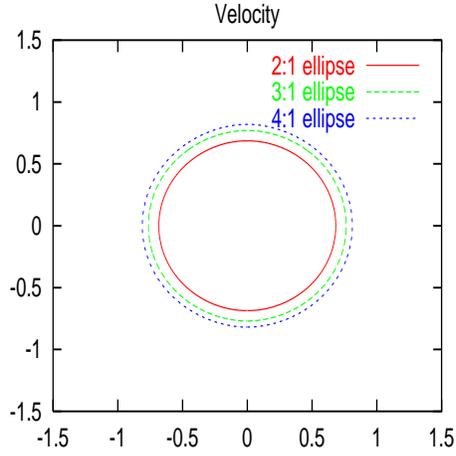


Figure 6: Velocity of the each point on elliptical boundaries of varying eccentricities.

Figure 7 shows the result of the time evolution for the ellipse. As time evolves the ellipse will rotate with a constant angular velocity while retaining its elliptical shape. This result is what one would expect from the analytic solution.

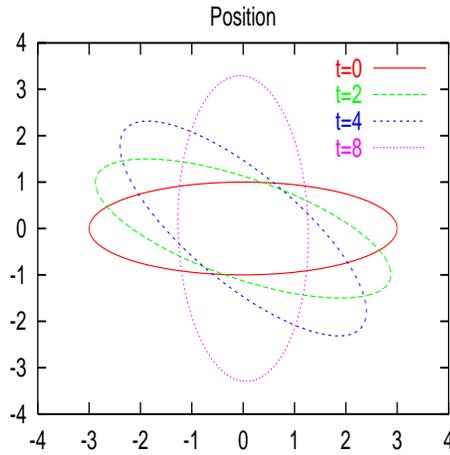


Figure 7: Position of the elliptical boundary at time t .

It is important to first study simple contours such as the circle and ellipse because the analytic solutions for these are known. Therefore, by arriving at the expected results for these shapes, one can be sure that the algorithm that is

used in the numeric calculations is working correctly. This way, when different contours are studied one may be confident that the results of the calculations are accurate.

This concludes the investigation that was done. The next step would be to study other shapes that rotate in time but are not deformed. Also of interest would be to study a similar system such as the two-dimensional electron gas in the presense of a magnetic field.

References

- [1] N. Zabusky, M. H. Hughes, and K. V. Roberts, "Contour Dynamics for the Euler Equations in Two Dimensions," *Journal of Computational Physics* **30**, 96 (1979).
- [2] C. Wexler and A. Dorsey, "Solitons on the Edge of a Two-Dimensional Electron System," *Phys. Rev. Lett.* **82**, 620 (1999).
- [3] I. G. Currie, *Fundamental Mechanics of Fluids*, p.13, McGraw-Hill: New York (1974).
- [4] P. Lions, *Mathematical Topics in Fluid Mechanics*, p.9-10, Oxford University Press: New York (1996).