

# Simulation and Analysis of Pattern Formation in 2-Dimensional Electron Gas Systems in the Quantum Hall Regime

Taylor Hughes  
University of Florida

July 31, 2002

## Abstract

It is shown that an inhomogeneous magnetic field surrounding a 2-dimensional electron gas droplet causes the droplet to form fingering patterns. The fingering patterns were produced by simulating the droplet using a Monte Carlo program. Further analysis was done on symmetric fingering cases in order to compare the electron droplet problem to a symmetric viscous fluid, radial fingering problem. A correspondence between the two analogous problems is constructed and then, since the fluid problem can be solved, its analytic solution is compared to the Monte Carlo data via the constructed correspondence. There is excellent agreement between the two analogs. In addition, since the analytic solutions, which are only approximations, form cups in a finite time it was possible to observe the physical behavior of the electron system at this critical state. In the first trials it appears that the droplet separates at this point, invalidating the simulation, and inhibiting further tests. However, new directions in the research might lead to better observations.

## 1 Introduction

Many phenomena in disparate branches of physics share similar features due to an underlying similarity in the equations that describe the phenomena. One such situation is that of the *Laplacian growth problem*. At the end of 19<sup>th</sup> century, physicists, most notably Hele-Shaw, observed strange fingering effects at the driven interface between two fluids. This surface instability was cast into a Laplacian growth problem by Saffman and Taylor in the 1950s [6]. Their work led to further detailed analysis of the fluid fingering problem as well as more careful experimental work.

The problem studied by these physicists focuses on the evolution of the boundary between two fluids of different viscosities in an effectively two dimensional domain. It can be shown that this *free boundary problem* is governed by equations for Laplacian growth [6]:

$$\nabla^2 p = 0 \quad \text{in fluid interior } D, \quad (1)$$

$$p = 0, \quad v_n = -\partial_n p \quad \text{on } \partial D, \quad (2)$$

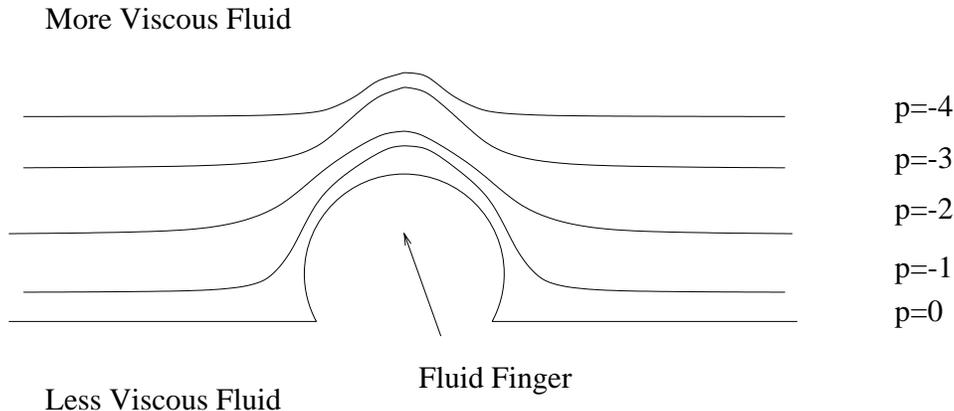


Figure 1: Schematic of the unstable growth of a fluid interface. Each curve represents an equipotential of the pressure. Since  $\mathbf{v} = -\nabla p$ , the interface moves fastest at the tip of the finger where the equipotentials are closest. This means that the tip will grow the fastest, followed closely by the rest of the finger.

$$p = -\frac{1}{2} \ln |z|, \quad |z| \rightarrow \infty. \quad (3)$$

Here Eq. (1) comes from Darcy’s Law,  $\mathbf{v} = -\nabla p$ , and the assumption that the fluid is incompressible,  $\nabla \cdot \mathbf{v} = 0$ . In these equations  $p$  is pressure,  $z$  is the coordinate in the complex plane, and  $v_n$  is the velocity normal to the fluid interface. The term Laplacian growth comes from this equation. The other two equations are boundary conditions for this differential equation. Since the problem satisfies Laplace’s equation and is a 2-dimensional problem, conformal mapping techniques are useful in solving it.

More recently, Agam *et al.* [1] predicted that quantum Hall (QH) droplets would exhibit these same fingering effects if placed in an inhomogeneous magnetic field. Agam *et al.* mapped the quantum Hall problem onto the mathematically similar fluid problem[1]. This means that the same equations model both of these analogous systems. It is these equations which lead to unstable fingering and suggest that the QH droplet should exhibit fingering behavior, when certain conditions are met.

In both cases the fingering arises because of instabilities in the boundary of the fluid domains or the electron domain. It is easiest to visualize by considering the fluid as shown in Fig. (1). When the less viscous fluid is pumped outward against the domain of the more-viscous fluid, the interface starts to become unstable. Due to small initial perturbations, wave-like oscillations occur on the boundary curve. The points on the interface with the greatest outward displacement move the fastest. This results in the largest oscillations growing into finger-like shapes[6]. Since the interface is unstable, the “fingers” continue to grow and propagate outward into the more viscous fluid with increasing velocity. This is why this type of problem is called a “growth” problem, and also why the phenomenon is referred to as “fingering.”

Along with the fingering there is the even more interesting prediction of cusp forma-

tion after a finite time. The cusp formation is an artifact of certain approximations used to simplify the equations. Without the consideration of stabilizing factors, which complicate the equations, the tips of the fingers steadily increase in velocity until they reach infinite velocity. A cusp forms in the boundary at the points where the fluid has infinite velocity. For the fluid problem, the surface tension that is ignored by the approximations naturally inhibits the formation of cusps. A semi-classical approximation used for the quantum droplet problem leads to the cusp formation. However, in a real system quantum mechanics would inhibit the cusps [1]. Thus far only behavior before the critical cusp formation time can be analyzed. However, this project should aid in the analysis of what actually happens at the critical time, and afterward.

System	Expanding Material	Growth Parameter	Stabilizing Factor	Conserved Moments
2DEG	Electrons	$N_e$	Quantum Mechanics	Conserved $t_k$
Fluid	Less-Viscous Fluid	Time ( $t$ )	Surface Tension	Conserved $C_k$

Table 1: Table listing analogous features of the 2-dimensional electron gas (2DEG) and viscous fluid problems.

## 2 Formulation of the Problem

### 2.1 Monte Carlo Simulation

Using Monte Carlo simulations, it is possible to analyze the probability density of an electron droplet. Subsequently, the boundary of the droplet can be extracted and compared with analytical solutions. As more electrons are added to the system, the droplet begins to finger radially outward. At a certain critical number of electrons, cusps form in the semi-classical solution which invalidate it. However, the Monte Carlo simulations embody the quantum mechanics of the droplet, except for the electron Coulomb interaction, and do not display cusps. This simulation allows us to expand the droplet to a state well beyond our current ability to predict, and to observe new phenomena associated with the droplet.

The general problem of an electron droplet surrounded by an inhomogeneous magnetic field is a hard one, but it can be simplified. Particularly, it can be solved if the inhomogeneity is produced by thin solenoids represented by delta functions, and if the Landé  $g$ -factor is assumed to be 2. The quantum Hall system we used had a filling fraction  $\nu = 1$ , and magnetic length  $\ell_b = \sqrt{\hbar c/eB_0}$ . Without the added inhomogeneous field terms, the ground state of the QH droplet is the wave function [4],

$$\Psi = \prod_{j < k}^{N_e} (z_j - z_k) \exp \left[ -\frac{1}{4\ell_b^2} \sum_{l=1}^{N_e} |z_l|^2 \right]. \quad (4)$$

Here  $\Psi$  is just a Slater determinant of single-particle wave functions since we have chosen  $\nu = 1$ . The probability density for the wave function,  $|\Psi|^2$ , can be interpreted as a Boltzmann weight  $e^{-\beta U}$ , where

$$U = - \sum_{i < j} \ln |z_i - z_j| + \frac{1}{4\ell_b^2} \sum_{j=1}^{N_e} |z_j|^2, \quad (5)$$

and  $\beta$  is arbitrarily set to two[4]. (Note: In our units  $\ell_b = \hbar = e = c = B_0 = 1$ .) This is the energy for a fictitious 2-dimensional, one-component classical plasma, and it can be employed in a Metropolis algorithm program to drive the system to its ground state. It should be noted that this is the energy for the fictitious plasma and is *not* related to the energy of the QH droplet.

Similarly, with the addition of the discrete solenoid fields, the inhomogeneous case can be converted into a classical 2-dimensional plasma problem. This time, however, there are additional terms in the energy that correspond to the solenoids and behave as impurity charges. This energy for the inhomogeneous system is:

$$U = - \sum_{i < j} \ln |z_i - z_j| + \frac{1}{4\ell_b^2} \sum_{j=1}^{N_e} |z_j|^2 - \sum_{a=1}^{N_s} \sum_{j=1}^{N_e} \frac{\Phi_a}{\Phi_0} \ln |z_j - z_a|. \quad (6)$$

Here,  $z_a$  is the complex position of the  $a$ th solenoid, and  $\Phi_a$  is the flux in the  $a$ th solenoid. The Monte Carlo program we wrote is used to drive the QH droplet system to its ground state and then to measure the properties of the state. Once the system has reached its minimum energy state the program averages over the electron positions to approximate the probability density of the droplet. Although the parameterized curve for the boundary cannot be determined, the points on the boundary can be extracted. The boundary of the probability distribution is important because it helps us to make comparisons between the numerical Monte Carlo simulation and the analytic solution for the fluid problem.

Choosing the boundary is a bit arbitrary since there are many ways to compute it. One method is to choose points of the distribution that lie within a certain probability range. This range is arbitrary and would naturally be picked to best fit the data to the theoretical predictions. Another method is to pick points that have the greatest negative probability gradient. This gets the points that are located where the distribution starts to fall off rapidly which gives a good representation of the boundary. Yet another way to extract the boundary is to use *contour*, which is one of Matlab's built in plotting functions. We tried all of these methods and obtained very good agreement among them. Therefore, we chose to work with the one that was easiest to use, the Matlab function. Using Matlab allowed us to quickly extract and plot the boundary contour from the probability distribution generated by our simulations.

We ran our Monte Carlo "experiments" with symmetric arrangements of solenoids placed outside of the QH droplet domain. More specifically, we placed solenoids equally spaced around a ring with a radius much greater than the characteristic radius of the droplet. The flux from these solenoids summed to zero to keep the system "charge"-neutral, and the flux alternated in sign around the ring. In a real physical system the impurity charges

could be distributed randomly, but we chose these symmetric cases for two reasons. First, symmetric arrangements gave us complete control of every aspect of the simulation. Second and most importantly, this symmetry made analytic comparisons with the fluid analog more tractable.

The link between the free-boundary fluid problem and the electron gas problem comes from Agam *et al.* [1]. It is known that the harmonic moments  $C_k$  of the viscous fluid region are conserved [5], where

$$C_k = -\frac{1}{\pi k} \int_{\text{fluid domain}} \frac{d^2z}{z^k}, \quad k = 1, 2, \dots \quad (7)$$

The harmonic moments of the QH droplet in an inhomogeneous magnetic field are conserved as well, implying that the droplet obeys the Laplacian growth equations [1]. It should be noted, however, that the inhomogeneities are not penetrating the droplet and remain in the outer domain. The field that penetrates the droplet is completely uniform since the extra solenoid contributions are confined to lie outside the droplet. Classically one would expect that these solenoids would not affect the droplet, but the droplet does exhibit fingering as it grows. This fingering is a completely quantum mechanical effect that cannot be explained with classical physics. It is the non-zero magnetic vector potential that affects the droplet's growth, through the Aharanov-Bohm effect [7].

## 2.2 Analytic Comparison with Fluid System

The boundary of the fluid interface is a very complicated curve that yields a difficult boundary value problem; but conformal mapping techniques simplify this. A useful function for mapping the exterior of the fluid in the  $z$ -plane to the exterior of the unit circle in the  $w$ -plane is [3]:

$$z(w, t) = \sum_{n=-1}^M a_n(t) w^{-n}. \quad (8)$$

This will be a solution for the Laplacian growth problem, provided the  $a_n(t)$  satisfy a set of differential equations [3]. The two cases we considered were those of an elliptical boundary and a boundary with  $D_6$  symmetry.

For the two special cases mentioned above we solved the simplified sets of differential equations for  $a_n(t)$ . To observe the behavior we wanted, we needed only two terms in the conformal map. So we used maps of the form:

$$z(w, t) = a(t)w + b(t)w^{-M}. \quad (9)$$

This map greatly simplifies the analysis. (Note: from now on the functional dependence of  $a$  and  $b$  on  $t$  will be dropped in the notation.)

The simpler of the two cases is the ellipse. The correct conformal map is Eq. (9) with  $M = 1$ , that is,  $z(w, t) = aw + bw^{-1}$ . The equations in Howison's paper [3] can be solved to yield solutions for the map parameters  $a$  and  $b$  in terms of the initial conditions. These

parameters evolve in time and produce a sequence of ellipses that grow with time. Here the major axis of the ellipse  $A$  is  $a + b$  and the minor axis  $B$  is  $a - b$ , where  $a$  and  $b$  are these same map parameters. It will be shown later that placing four solenoids on a ring around the electron droplet causes it to grow as an ellipse [8].

For the second case we have a fluid domain that has  $D_6$  symmetry (*i.e.*, it looks like a triangle with curved sides). The conformal map here is  $z(w, t) = aw + bw^{-2}$ , which is just Eq. (9) with  $M = 2$ . As with the previous example, we solved differential equations for  $a(t)$  and  $b(t)$ . This fluid system corresponds to a QH droplet system with six solenoids.

As noted before, a correspondence exists between the fluid problem and the QH droplet problem. This correspondence is built on the harmonic moments that are conserved for each system. Since the moments are conserved, they can serve as initial data for these systems. This means that if moments are specified for one system and the other system is given the corresponding moments, then the two systems will evolve identically. For the fluid system we have the moments given in Eq. (7) and for the QH system we have:

$$t_k = \frac{1}{\pi k} \int \frac{\delta B}{B_0} z^{-k} d^2 z = \frac{2}{k} \sum_{n=1}^{N_s} \frac{\Phi_n}{\Phi_0} z_n^{-k}, \quad (10)$$

with  $\delta B = \sum_n^{N_s} \Phi_n \delta^{(2)}(z - z_n)$ . The correspondence that maps variables between the two systems is:  $C_k = \pi k t_k$  [1].

It turns out that the most important moments for our chosen conformal map are  $C_{M+1}$  and  $C_0$ . Many of the moments for  $k > 0$  are zero, and  $C_{M+1}$  is the greatest of those that are nonzero. Ignoring the higher moments is allowable in our experiments since each greater moment is smaller by a factor of  $1/R$ , where  $R$  is the distance from the origin to the ring of solenoids. We know that  $C_0$  is just equal to the area of the quantum droplet, which in terms of the map parameters means:

$$C_0 = \pi(a^2 - Mb^2) = 2\pi N_e. \quad (11)$$

For  $C_{M+1}$  we have:

$$C_{M+1} = \frac{\pi b}{a^M} = \frac{4\pi(M+1)\Phi_n}{R^{M+1}}. \quad (12)$$

What we are looking for is a way to tune the fluid analytic solution to the parameters of the Monte Carlo simulation. This means that we need to invert these last two equations. This will give us conformal mapping parameters  $a$  and  $b$  for each value of the solenoid flux  $\Phi$ , distance to the solenoids  $R$ , and number of electrons  $N_e$ . Solving Eqs. (11) and (12) with arbitrary  $M$  we obtain:

$$a^{2M} - \left( \frac{\pi^2}{MC_{M+1}^2} \right) a^2 + \frac{\pi C_0}{MC_{M+1}^2} = 0, \quad (13)$$

$$b = \sqrt{\frac{\pi a^2 - C_0}{M\pi}}. \quad (14)$$

In general these equations are not easy to solve since Eq. (13) is a  $M$ th degree polynomial in  $a^2$ , but some simplifications can be made. If we assume that  $a$  and  $b$  are real, and that  $a$  is positive, then using Descartes rule of signs there exist either two or zero solutions for  $a$ . Though this does simplify our pursuit it does not suggest a general way to solve these equations. Fortunately they can be solved for the two cases we are interested in:  $M = 1$ , and  $M = 2$ .

First let us consider the simpler  $M = 1$  case. By substituting  $M = 1$  into Eq. (13) and Eq. (14) we can solve for  $a$  and  $b$  to get:

$$a = \sqrt{\frac{C_0}{\pi(1 - C_2^2)}}, \quad (15)$$

$$b = aC_2. \quad (16)$$

By way of Eqs. (11)-(12) we now have made the conformal map a function of the QH droplet parameters. This conformal mapping produces an ellipse as the fluid interface. The elliptical boundary is stable and never forms cusps[2]. This means that our simulations should match up with the analytic solution as long as the boundary is far from the solenoids and  $|C_2| < 1$ . Following the same procedure for  $M = 2$  yields:

$$a = \frac{1}{2C_3} \left[ \pi^2 - \sqrt{\pi^4 - 8\pi C_0 C_3^2} \right]^{\frac{1}{2}}, \quad (17)$$

$$b = \frac{1}{4\pi C_3} \left[ \pi^2 - \sqrt{\pi^4 - 8\pi C_0 C_3^2} \right]. \quad (18)$$

Since we have  $M = 2$  there are actually four possible solutions for  $a$ . However, with the assumptions above, and the added assumption that we eventually want cusps to form, Eq. (17) is the solution. This mapping produces an interface with  $D_6$  symmetry, and the solution is unstable. Eventually, once the fluid has evolved outward, cusps will form in the solution. This critical time at which the cusps form in the fluid corresponds to a critical number of electrons in the QH droplet.

We are now in a position to find this critical number of electrons  $N_e^*$ . The conformal map has cusps when it is no longer conformal. This condition is implied by its derivative equaling zero. Setting  $dz/dw = 0$ , where  $z$  is from Eq. (9), we get the condition that cusps form when

$$\frac{Mb}{a} = 1. \quad (19)$$

Using this equation to eliminate  $b$ , then using Eq. (12) to eliminate  $a$ , we can solve Eq. (11) for  $N_e^*$ . This gives us

$$N_e^* = \frac{M - 1}{2M} \left( \frac{\pi}{MC_{M+1}} \right)^{\frac{2}{M-1}}. \quad (20)$$

With these results we were able to make detailed comparisons between the two analogous problems.

### 3 Results and Conclusions

The plots in Figs. (2)-(3) are for an evolving elliptical boundary, and the plots in Figs. (5)-(6) are for an evolving boundary with  $D_6$  symmetry. The thin lines are the boundaries of the Monte Carlo simulation droplet, and the bold dots show the analytic solution. Notice here that in the first set of plots no cusps form, but in the second set, the boundary of the distribution is clearly forming cusps.

In the first set of plots with the elliptical boundaries, the data match the analytical solution very well. So, with four solenoids set up as mentioned before, the QH droplet will grow outward with an elliptical boundary. This makes sense because two of the solenoids are pushing on the droplet, and two are pulling. This compresses one axis of symmetry and expands the other, which means a droplet that started with a circular boundary would be transformed into one with an elliptical boundary. As more electrons enter the system the droplet gets larger, but the elliptical boundary remains stable, and has a constant eccentricity, provided the droplet remains far from the solenoids[2].

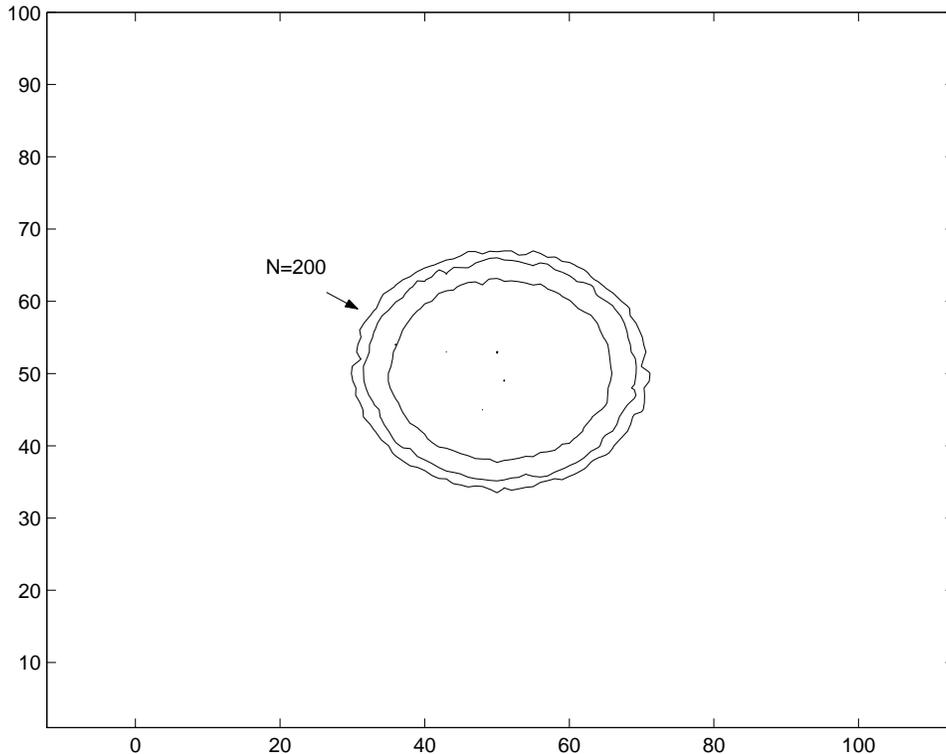


Figure 2: Boundary of the simulated QH droplet surrounded by four solenoids for values of  $N_e = 100, 175, 200$ . The length scale is in units of magnetic length  $\ell_b$ . The solenoids are on a ring with  $R = 200$  and center  $(50, 50)$ .

In the second set of plots, the data also match the analytical solution well. In this

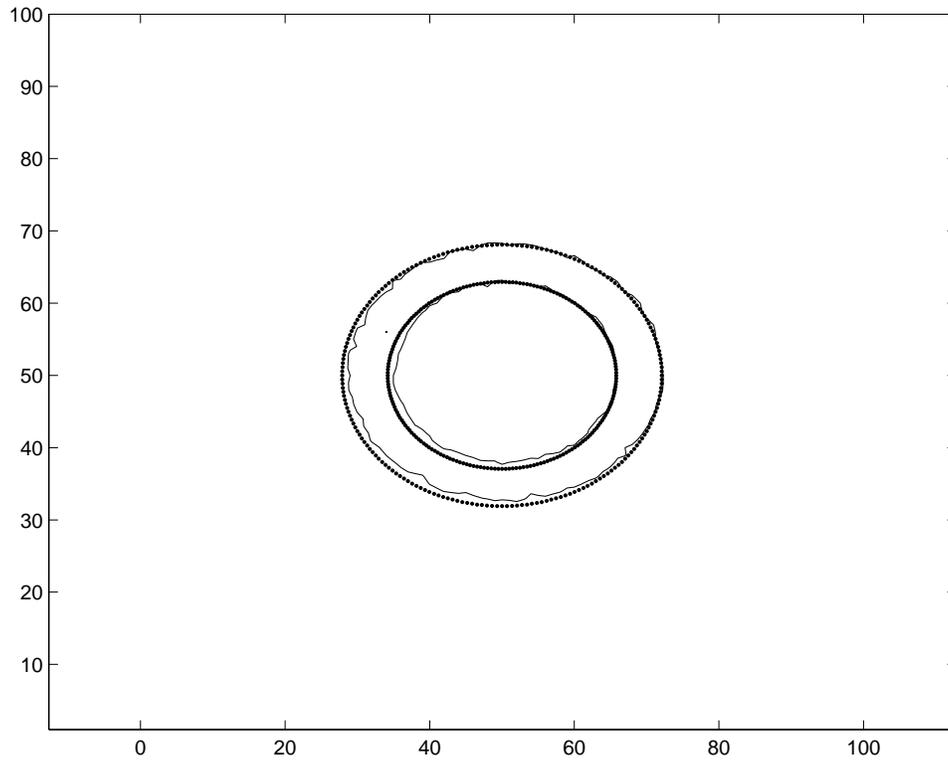


Figure 3: Analytic fluid boundary superimposed on the Monte Carlo data for  $N_e = 100, 200$ . The bold dotted line is the analytic solution, and the thin line is the Monte Carlo simulation.

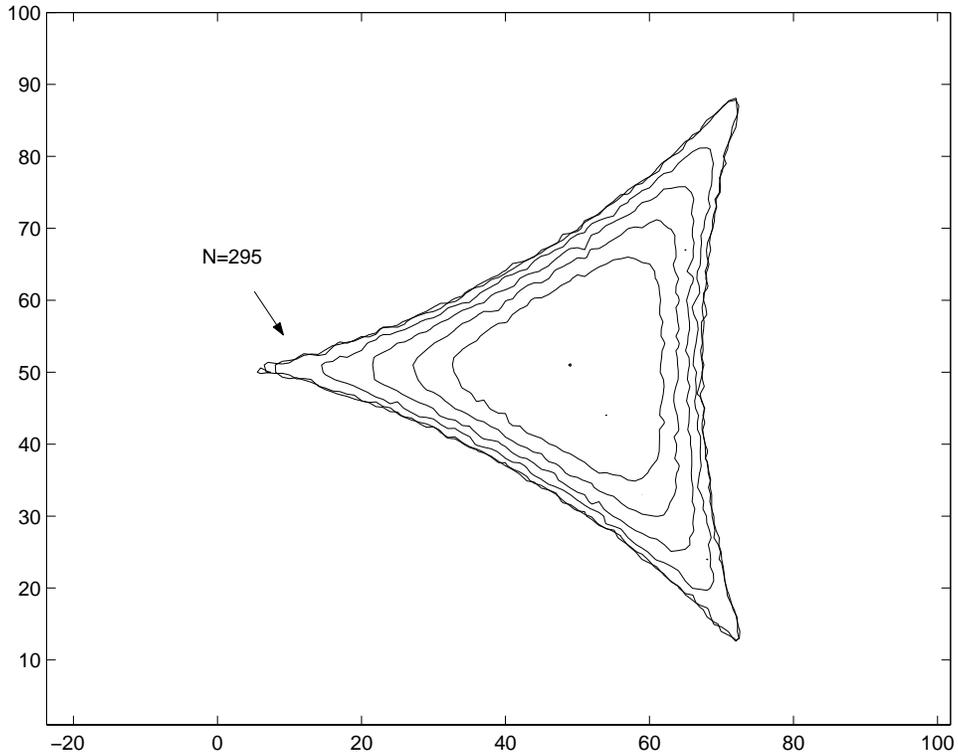


Figure 4: Boundary of the simulated electron droplet surrounded by six solenoids for  $N_e = 100, 150, 200, 250, 290, 295$ . The electron broke apart at  $N_e = 296$ , and it had a predicted  $N_e^* = 299$ . The length scale is in units of magnetic length  $\ell_b$  and the solenoids are on a ring with  $R = 200$  and center  $(50, 50)$ .

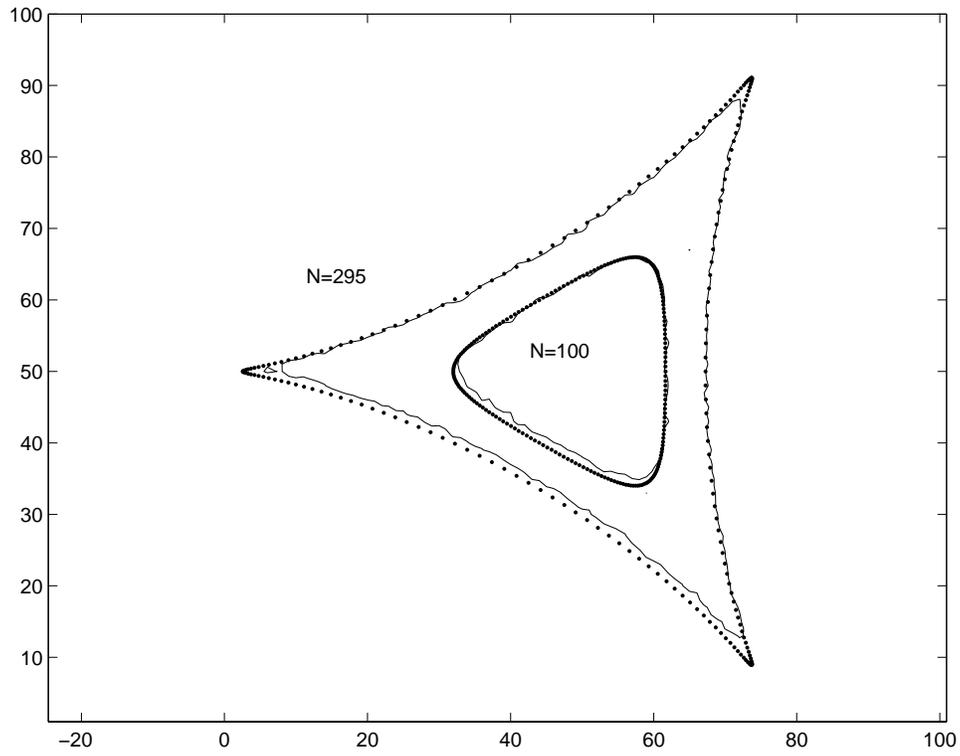


Figure 5: Analytic fluid boundary superimposed on the Monte Carlo data for  $N_e = 100, 295$ . The bold dotted line is the analytic solution, and the thin line is the Monte Carlo simulation.

case we have six solenoids. Three of these are repelling the system to make the boundary curve inward. The other three attract the charges and help to create the long fingers. The initial conditions specify that  $N_e^* = 299$ . This means that once 299 electrons have been added to the droplet, we should see the behavior that occurs in place of the cusp formation. Indeed, in the simulations we find that the droplet fingers as the analytic solution predicts, but with slightly more rounded features. Then, once the QH droplet reaches 296 electrons, the droplet becomes unstable and breaks apart. So when the analytic solution forms cusps, the simulated droplet breaks into three pieces and moves out to the solenoids. After the droplet breaks apart our simulation is invalid because the energy of the system tends to negative infinity, so behavior after the break up cannot be examined.

In all of the plots there is excellent agreement between the analytic solutions and the Monte Carlo data. This agreement supports the prediction made by Agam *et al.* [1] that the QH droplet will display fingering behavior. Also, the approximations that we made with the conformal maps seem to be well suited since there was such a close match. So now, in place of the cusp singularities, we have seen that the QH droplet becomes unstable and breaks apart. We cannot be explicit as to what happens after the droplet becomes separated, but at least we have an idea now of its behavior.

From this point there are several new research directions that can be taken. The Monte Carlo program is robust enough to adapt to almost any interesting droplet configurations, and would allow us to examine the case with one solenoid on the ring, which does have a solvable fluid counterpart. It will also be interesting to observe the phenomena resulting from two or more separate droplets that will grow and finger on their own. The next step, however, will be to replace the solenoids with a potential that is very similar, but will not break the droplet apart. This last direction should give us a good idea about behavior at the critical time, and how the droplet evolves afterward.

## 4 Acknowledgments

I would like to thank my collaborators Alan T. Dorsey and Alexios Klironomos whose work made this project possible. They share the credit with me for all of the research completed. I would also like to thank Filippos Klironomos for helpful insight. Finally, I would like to thank the NSF and the University of Florida REU program for funding my research.

## References

- [1] Oded Agam, Eldad Bettelheim, P. B. Wiegmann, and A. Zabrodin. Viscous fingering and the shape of an electronic droplet in the quantum hall regime. *Phys. Rev. Lett.*, 88:236801, 2002.
- [2] S. D. Howison. Bubble growth in porous media and hele-shaw cells. *Proceedings of the Royal Society of Edinburgh*, 102A:141, 1986.

- [3] S. D. Howison. Cusp development in hele-shaw flow with a free surface. *SIAM J. Appl. Math.*, 46:20, 1986.
- [4] R.B. Laughlin. *Phys. Rev. Lett.*, 50:1395, 1983.
- [5] S. Richardson. Hele-shaw flows with a free boundary produced by the injection of fluid into a narrow channel. *J. Fluid Mech.*, 56:609, 1972.
- [6] P. G. Saffman and G. I. Taylor. *Proc. R. Soc. London, Ser. A*, 245:2312, 1958.
- [7] P. B. Wiegmann. Aharonov-bohm effect in the quantum hall regime and fingering instability. *cond-mat/0204254*.
- [8] P. B. Wiegmann and A. Zabrodin. Conformal maps and integrable hierarchies. *Commun. Math. Phys.*, 213:523, 2000.