Hydrodynamics and turbulence in classical and quantum fluids

Introduction to Turbulence
Turbulence is widespread, indeed almost the rule, in the flow of fluids. It is a complex phenomenon, for which the development of a satisfactory theoretical framework has been one of the greatest unsolved challenges of classical physics.

The first scientific investigations of fluid turbulence are generally attributed to Leonardo Da Vinci.
A few of the Nobel laureates who at some point tangled with the turbulence problem.…

Onsager: Chemistry 1968 (thermodynamics of irreversible processes)

Feynman: Physics 1965 (quantum electrodynamics)

Wilson: Physics 1982 (critical phenomena/renormalization group)
In 1963 Richard Feynman wrote*: 

“Finally, there is a physical problem that is common to many fields, that is very old, and that has not been solved. It is not the problem of finding new fundamental particles, but something left over from a long time ago – over a hundred years. Nobody in physics has really been able to analyze it mathematically satisfactorily in spite of its importance to the sister sciences. It is the analysis of circulating or turbulent fluids.”

*Thanks to Eberhard Bodenshatz
Turbulence exists in a wide range of contexts such as the motion of submarines, ships and aircraft, pollutant dispersion in the earth's atmosphere and oceans, heat and mass transport in engineering applications as well as geophysics and astrophysics.

The problem is also a paradigm for strongly nonlinear systems, distinguished by strong fluctuations and strong coupling among a large number of degrees of freedom. [G. Falkovich, K.R. Sreenivasan, *Phy. Today* 59, 43 (2006)].

Turbulence is particularly useful because the equations of motion are known exactly and can be simulated with precision. And so, even distant areas such as fracture [M.P. Marder, *Condensed Matter Physics*. Wiley, New York (2000)]---perhaps even market fluctuations [B.B. Mandelbrot, *Scientific American* 280, 50 (1999)]---may benefit from a better understanding of it.
The complexity of the underlying equations has precluded much analytical progress, and the demands of computing power are such that routine simulations of large turbulent flows has not yet been possible.

Thus, the progress in the field has depended heavily on experimental input. This experimental input in turn points in part to a search for optimal test fluids, and the development and utilization of novel instrumentation.

The question we will ask is whether cryogenics can get us closer to an understanding of these and other phenomena.
Some defining characteristics of turbulence *

**Irregularity:** Turbulent flows are irregular and random. This complexity will exist in both space and time (spatial irregularity itself clearly does not constitute turbulence nor does the converse). Even though the deterministic Navier Stokes equations presumably contain all of turbulence it is impossible to predict the precise values of any variables at any time.

Statistical measures, however, are reproducible and this has led to statistical approaches toward solving the NS equations. This always leads to a situation in which there are more unknowns than equations—the so-called closure problem.

**Diffusivity:** The most important aspect of turbulence as far as applications are concerned is its associated strong mixing and high rates of momentum, heat and mass transfer. The randomness or irregularity is not sufficient to define turbulence—turbulent flows will always exhibit strong spreading of fluctuations.

*reference: H. Tennekes and J. Lumley*
**High Re:** Turbulent flows exist only for high values of Re. They often originate from instabilities in the fluid such as those in RB convection that we looked at in the last lecture.

**Dissipation:** If no energy is supplied turbulence will decay rapidly. It needs to acquire energy from its environment. We will look at decaying turbulence in the quantum context in lecture 5.

**Stretching:** Turbulence must then be maintained and vortex stretching is an important process. 2D flows cannot have vortex stretching, and so large scale atmospheric cyclonic flows, which are essentially 2D, are not usually considered in themselves turbulent in the sense that we will describe later, although their characteristics are strongly influenced by the turbulence generated through buoyancy and shear.

We will have to consider how vortex stretching can occur in quantum turbulence where we have a restriction of a single quantum of vorticity for each vortex filament! It can and we will see how in lecture 5.
Flows: Turbulence is a property of the flow not the fluid (although it is tempting to find effective viscosities or diffusivities that represent the enhanced transport of turbulent flows).

Mathematically, the details of the transition to turbulence remain poorly understood. Most of the theory of hydrodynamic instabilities in laminar flow is linearized theory as we discussed yesterday and hence valid only for small disturbances. Presumably, instabilities have secondary instabilities and so on creating “lumps” of turbulence which then merge with others to produce a turbulent flow.
\[ \frac{\partial v}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} \]

\[ \nu \frac{\mathbf{v} \cdot \nabla \mathbf{v}}{\nu \nabla^2 \mathbf{v}} \sim \frac{vL}{\nu} = \text{Re} \]

Reynolds

This interpretation of Re is not so straightforward since at high Re viscous effects operate at different length scales than do inertial effects.

Some dimensional considerations

The molecular time scale for momentum diffusion is (dimensionally):

$$t_M \sim \frac{L^2}{\nu}$$

A characteristic time scale for turbulent flows can be estimated from the largest scale flows, of characteristic length $L$ and velocity $u$, which are the most effective at mixing (again dimensionally):

$$t_T \sim \frac{L}{u}$$

Then

$$\frac{t_M}{t_T} \sim \frac{uL}{\nu} = \text{Re}_L$$

$\text{Re}_L$ of a turbulent flow then can be interpreted as the ratio of a characteristic molecular time scale to a turbulent time scale in the case that the former is evaluated over the same length scale. For large $\text{Re}_L$ (the large scales) viscosity is unimportant (as we have seen before).
It is tempting, but not necessarily useful to think in terms of an effective “eddy” viscosity or diffusivity \( K \) so that there is a simple diffusion time, say for heat, for our effective “fluid”

\[
\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x_i \partial x_i}
\]

\( t_{\text{turbdiff}} \sim \frac{L^2}{K} \)

Setting this turbulent diffusion time equal to the actual turbulent scale we have (for diffusion of heat)

\[
\frac{L^2}{K} = \frac{L}{u}
\rightarrow K = uL
\]

\[
\frac{K}{\nu} \approx \frac{K}{\nu} \sim \frac{uL}{\nu} = \text{Re}
\]

(Valid for gases—we will look at this later experimentally)

So that \( \text{Re} \) also appears as a ratio of this turbulent diffusion to molecular diffusion and reinforces the statement that turbulent flows are characterized by strong diffusion or spreading of disturbances.
Eddies produced by flow (e.g., through a grid at length scale $M =$ mesh size).

Cascade of energy in inertial range (local transfer of energy due to nonlinear inertial term in NS eqn.).

Viscous dissipation at small scales for which local $Re \sim 1$.

(Aproximately) homogeneous turbulence

Energy Cascade

$E(k) = C \varepsilon^{2/3} k^{-5/3}$
More on small scales and the energy cascade…

Reducing viscosity (or increasing Re) does not alter the rate of energy dissipation per unit mass, $\varepsilon$, (which is determined from the energy input) but rather allows the cascade of energy to *continue to smaller scales*.

The smallest scale in turbulent flows is the dissipation length scale $\eta$, where viscosity becomes dominant. To see this we consider that the rate of energy supplied at the injection scale $L$ per unit mass is given by

$$u^2 \cdot u / L = u^3 / L$$

This energy is dissipated at a rate per unit mass, $\varepsilon$, which must be the same, i.e:

$$\varepsilon = u^3 / L$$

This must be true at *all* scales $\ell$ ($\varepsilon$ is a constant) and so we have in general

$$u_\ell \sim \varepsilon^{1/3} \ell^{1/3}$$
We can now write down the eddy turn-over time for the scale $\ell$ in terms of the energy dissipation rate:

$$t_\ell \equiv \frac{\ell}{u_\ell} \sim \varepsilon^{-1/3} \ell^{2/3}$$

We can also write down the viscous diffusion time $t_\ell^{\text{diff}}$ for the same scale $\ell$:

$$t_\ell^{\text{diff}} \sim \frac{\ell^2}{\nu}$$

Note that the diffusion time goes to zero faster with $\ell$ than does the eddy turnover time. Viscosity becomes important at a dissipation length scale $\ell_{\text{diss}}$ for which the two time scales are equal, i.e., for

$$\frac{\ell_{\text{diss}}^2}{\nu} = \varepsilon^{-1/3} \ell_{\text{diss}}^{2/3}$$
The separation between the energy-injection scales and the dissipative scales increases with Re.

So if there are any universal statistical properties of turbulence, it is reasonable to look for them as Re→∞

But… recall that Re→∞ is not the same as Re=∞ because in the former there exists a scale at which viscosity acts (its wavenumber may also go to infinity) while for the latter we have an ideal frictionless fluid at ALL scales not just an approximation for the large ones.

Denoting this dissipation scale $\ell_{\text{diss}}$ as $\eta$ (customary notation), we find

$$\eta = \left( \frac{V^3}{\varepsilon} \right)^{1/4} = L\text{Re}^{-3/4}$$
The difference between two flows with the same integral scale but different Re is the size of the smallest eddies. Index of refraction gradients are steep for the smallest eddies and hence shimmering seen on hot days.

A turbulent jet

“inner scales” of length, time, and velocity:

Scale relations:

\[ \eta \equiv \left( \frac{3}{\varepsilon} \right)^{1/4} \quad \frac{\eta}{L} \sim (uL/\nu)^{-3/4} = Re^{-3/4} \]

\[ \tau \equiv \left( \frac{3}{\varepsilon} \right)^{1/2} \quad \frac{\tau}{t_T} \sim (uL/\nu)^{-1/2} = Re^{-1/2} \]

\[ \frac{u_\eta}{u} \equiv \left( \frac{3}{\varepsilon} \right)^{-1/4} \quad \frac{u_\eta}{u} \sim (uL/\nu)^{-1/4} = Re^{-1/4} \]

Note that \[ \frac{\eta u_\eta}{\nu} = 1 \]

(a) “Low” Re. (b) “High” Re
Successive eddies have size

\[ \ell_n = Lr^n, \]

\((n = 0, 1, 2, \ldots)\)

\(0 < r < 1\)

Exact value has no meaning so we can take \(r = 1/2\)

To fill space, the number of eddies per unit volume is assumed to grow with \(n\) as \(r^{-3n}\)

This leads to two assumptions:

1) **Scale invariance within the inertial range.** This would be violated if the eddies were less and less space filling at small scales (in fact they are and this leads to corrections of the Kolmogorov theory.

2) **Localness of interactions:** energy flux involves scales of comparable size
Conceptual proof

Eddies of size $\ell$ will be swept along by eddies of size $L \gg \ell$. But this cannot affect the energy due to Galilean invariance of the NS equations. We can think of these processes as random Galilean transformations.

Consider also distortion: distortion is caused by shear, i.e. by velocity gradients. The typical shear associated with scales of size $\ell$ is:

$$s_\ell \sim \frac{u_\ell}{\ell} \sim \mathcal{E}^{1/3} \ell^{-2/3}$$

We see that the smallest shear is at the top of the inertial range and the greatest at the bottom (small scales). Note this latter fact is what keeps the dissipation term in the NS equations from going to zero as Re become large.

Therefore the shearing of an eddy of size $\ell$ by one of size $L \gg \ell$ is ineffective in producing distortions because there is little shear at those scales. Conversely, the shearing of an eddy of size $\ell$ by another of size $\ell' \ll \ell$ is also ineffective because the smaller eddy does not act coherently over the scale $\ell$. 
Hence: a \textit{cascade} of energy from \textit{scale to scale}

Dimensional analysis gives:

\[ E(k) = C \varepsilon^{2/3} k^{-5/3} \]

C is a constant of order 1 (from measurements it is estimated to be roughly 1.5)

Most of the energy is in the large scales while most of the vorticity is in the small scales.

An example of a “cascade” with a drop of dense ink in water
Importance of the local interaction picture

We may consider in certain circumstances that the characteristics of turbulence are controlled by the local environment; i.e. that local inputs of energy approximately balance local losses so that we have a “dynamical equilibrium.”

If energy transfer is sufficiently rapid that effects of past events do not dominate the dynamics this local equilibrium should be governed by local parameters such as scale lengths and times.

Scaling laws—both in spatial (time) and spectral (frequency) domain are at the heart of turbulence research.

Turbulence is characterized by the logarithm of Re so we need to generate orders of magnitude variations of Re in order to resolve scaling—this has always been a problem for experiments and, as we shall see, is one of the advantages of going to a low temperature environment and using helium as a test fluid.
Q: As $\text{Re} \to \infty$ can we neglect viscosity and recover the simpler Euler equations?

A: No, turbulence will always create small enough scales for viscosity to dominate.

Q: Will the continuum approximation remain valid as $\text{Re}$ increase more and more?

A: good question

Molecular and turbulent scales

Let us consider gases: on a molecular level the characteristic length is the mean free path $\xi$ and the velocity scale is the speed of sound $a$. The kinematic viscosity is approximated by:

$$\nu \sim a \xi$$
\[
\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} = \left(\frac{\nu^4 L u}{u^4 \nu}\right) = \left(\frac{\nu}{u}\right)\left(\frac{u L}{\nu}\right)^{1/4} = \left(\frac{\nu}{u}\right) \text{Re}^{1/4}
\]

Mean free path/ turbulent dissipation scale:
\[
\frac{\xi}{\eta} = \frac{\nu}{a} \cdot \frac{u}{\nu} \cdot \text{Re}^{-1/4} = \frac{M}{\text{Re}^{1/4}}
\]

\[M = \text{Mach number. High M and low Re is an unlikely combination.}\]

Within the Kolmogorov framework, the fluid is incompressible (M \(<< 1\)) so the higher the Re the better the continuum approximation! Of course if compressibility becomes important then the hydrodynamic approximation may come into doubt. This (compressibility + high Re) can happen in some astrophysical systems.
Reynolds –averaged equations

Recall:

\[ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \sigma_{ij} \]  \hspace{1cm} \text{General momentum equation}

\[ \frac{\partial u_i}{\partial x_i} = 0 \]  \hspace{1cm} \text{Conservation of mass}

\[ \sigma_{ij} = -p \delta_{ij} + 2 \mu s_{ij} \]  \hspace{1cm} \text{Stress tensor}

\[ s_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]
NS equations then followed (here without an external body force):

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

We now consider a decomposition of the velocity into a mean and fluctuation part:

$$u_i = U_i + u'_i$$

$$U_i = \lim(T \to \infty) \frac{1}{T} \int_0^{+T} u_i dt$$

$$\bar{u'_i} \equiv 0$$

Integrals have to be independent of starting time $t_0$ — assume stationarity
After some work we obtain an equation for the mean flow:

\[
\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left( \mu \frac{\partial U_i}{\partial x_j} - \rho u'_i u'_j \right)
\]

\[\tau_{ij} \equiv \rho u'_i u'_j\]  Reynolds stress tensor

3 diagonal components: \[\rho u'_1^2, \rho u'_2^2, \rho u'_3^2\]  Normal stresses or pressures

The off diagonal elements are the shear stresses and play dominant role in mean momentum transfer by turbulent motion. They represent correlations. This transfer –which is to cause *vorticity stretching* (under conservation of angular momentum) --drives the energy cascade.
Order of indices does not matter (the stress tensor is symmetric) and so there are 6 independent variables in all associated with the Reynolds stress tensor in addition to three for the mean velocity, and the pressure.

We have 4 independent equations (three components of the Reynolds momentum equation and the continuity equation

Problem! (This is usually referred to as the “Closure Problem” of turbulence.)
Alternatives to laboratory experiments

Could nature’s turbulence “laboratories”, such as the atmosphere and oceans, be instrumented and studied?

Yes…and they are. But this is not a substitute for controlled laboratory experiments, especially when questions become more refined. Boundary conditions, stationarity, need to be considered.

Alternatives to both “actual” experiments and theory

Direct numerical simulations (DNS) can be performed in which the appropriate equations are solved on a computer without making any approximation. The range of scales needing to be well resolved, however, grows rapidly as $Re^{3/4}$, and thus $Re^{9/4}$ in 3 dimensions. The state of the art in DNS is about $Re \sim 10^4$, or about 3-4 orders of magnitude lower than the $Re$ corresponding to a typically commercial jet aircraft, and the same amount for most atmospheric and oceanic flows.

Large eddy simulations (LES)—which compute only the large scales and model the small scales—do better, but they are not satisfactory for every problem.
Revisit of RB Convection

Rayleigh number: \[ Ra = \frac{g \alpha \Delta TH^3}{v \kappa} \]

Prandtl number: \[ Pr = \frac{v}{\kappa} \]

Aspect ratio: \[ \Gamma = \frac{D}{H} \]

\[ Nu = \frac{QH}{k \Delta T} \]
Some of the motivating examples of thermal convection at limiting values of the control parameters in nature. How to get there AND have the possibility to discern scaling relations? (next lecture)

- **Sun**
  - \( \text{Ra} \sim 10^{22} \)
  - \( 10^{-3} < \text{Pr} < 10^{-10} \)

- **Atmosphere**
  - \( \text{Ra} \sim 10^{17} \)
  - \( \text{Pr} \sim 0.7 \)

- **Outer core**
  - Thermal \( \text{Ra} > 10^{12} \)
  - \( \text{Pr} \sim 0.1 \)
  - Compositional \( \text{Ra} > 10^{15} \)
  - \( \text{Pr} \sim 100 \)
  - [exponents much larger if molecular properties used.]

- **Mantle**
  - \( \text{Ra} \sim 10^6 \)
  - \( \text{Pr} \sim 10^{21} \)
  - \( \text{Pr} \sim 10^3, \text{magma} \)
Measuring an eddy diffusivity

\[ \langle T_B \rangle + T_{B0} \cos(\omega t) \]

Thermal wave

Small sensor at mid-height

Amplitude decay ("skin effect")

\[ T_M(\zeta, \omega) = T_{B0} \text{rms} \exp(-z/\delta_S) \]

\[ \delta_S = \sqrt{\frac{2K_{\text{eff}}}{\omega}} \]

We find:

\[ \frac{K_{\text{eff}}}{K} \sim \text{Re} \]

Although this needs some further qualification in lecture 4
Irregular reversals of the large scale self-organized eddy
(Lecture 4)

Turbulent convection in the lab

From simulations

Geomagnetic polarity reversals: range of time scales ~ $10^3$-$10^5$ years.

Glatzmaier, Coe, Hongre and Roberts Nature 401, p. 885-890, 1999
Magnetic field of the Earth

The magnetic field is generated from turbulent convection within the Earth’s outer core. The main ingredients necessary appear to be a conducting fluid in turbulent motion (homogeneous and isotropic) interacting with the Coriolis force and giving rise to “order out of chaos.” As H.K. Moffatt says in rhyme:

Convection and diffusion
In turb’lence with helicity
Yields order from confusion
In cosmic electricity!