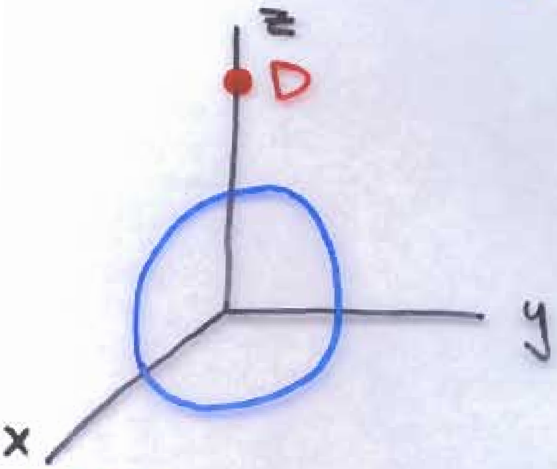
Helping Newton with Math!

* unit vector pointing from $\vec{x}_1$ to $\vec{x}_2 = \dfrac{\vec{x}_2 - \vec{x}_1}{\|\vec{x}_2 - \vec{x}_1\|}$
* sphere of radius R, density $\rho_0 = \dfrac{M}{\frac{4}{3}\pi R^3}$
* Force at $\vec{x}_1 = D\hat{k}$ from $\vec{x}_2 = (x, y, z)$
* $dM = \rho_0\, dx\,dy\,dz \Rightarrow d\vec{F} = \dfrac{G\,dM\,m}{\|\vec{x} - D\hat{k}\|^2} \dfrac{(\vec{x} - D\hat{k})}{\|\vec{x} - D\hat{k}\|}$



$$\vec{F} = \int_{\text{sphere}} d\vec{F}$$

① Change from x & y to $\left\{ \begin{array}{l} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}(y/x) \end{array} \right\}$

$$\vec{F} = G\rho_0 m \hat{k} \int_{-R}^{R} dz \int_0^{\sqrt{R^2-z^2}} dr\, r \int_0^{2\pi} d\theta \frac{z-D}{\left[r^2 + (z-D)^2\right]^{3/2}}$$

② Perform θ & r integrals

$$\vec{F} = G\rho_0 m \hat{k}\, 2\pi \int_{-R}^{R} dz \left\{ \frac{z-D}{|z-D|} - \frac{(z-D)}{\sqrt{R^2 - 2zD + D^2}} \right\}$$

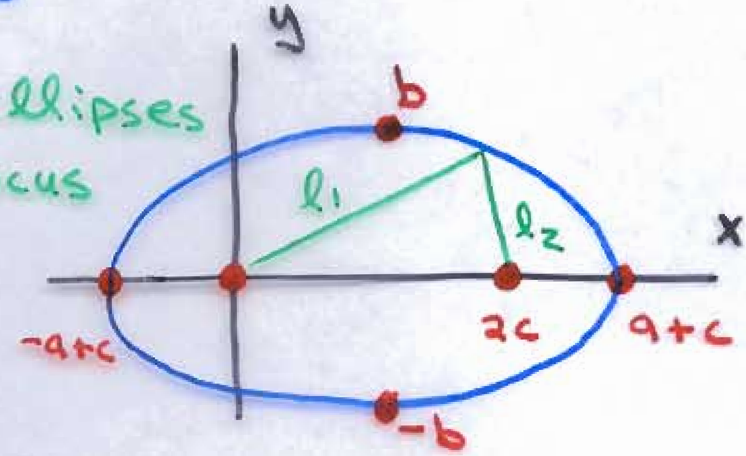
③ $D > R \Rightarrow \vec{F} = -\dfrac{GMm}{D^2}\hat{k}$

④ $D < R \Rightarrow \vec{F} = -\dfrac{GMm}{R^2}\dfrac{D}{R}\hat{k}$

Kepler's Laws

① Law of orbits

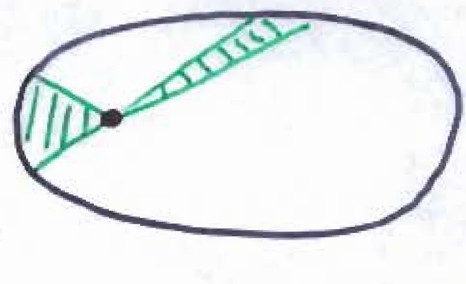
Planets orbit on ellipses with Sun at one focus



* semi-major axis a
* semi-minor axis b
* $l_1 + l_2 = 2a$
* eccentricity $\epsilon = \frac{c}{a}$ where $c^2 \equiv a^2 - b^2$

② Law of Areas

Equal areas swept out in equal times



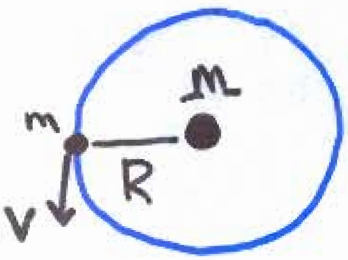
* follows from $\frac{dL}{dt} = 0$

$$dA = \frac{1}{2} \cdot r \cdot r\,d\theta = \frac{L}{2m}\,dt \Rightarrow \frac{dA}{dt} = \frac{L}{2m}$$

③ Law of Periods

$T^2 = \text{const} \cdot a^3$ for all planets

* special case of $\epsilon = 0 \Rightarrow c = 0$ & $a = b = R$



$$m\frac{v^2}{R} = \frac{GMm}{R^2} \quad \& \quad v = \frac{2\pi R}{T}$$

$$\therefore T^2 = \frac{4\pi^2}{GM} R^3$$

This is how we weigh planets!

11/10/14
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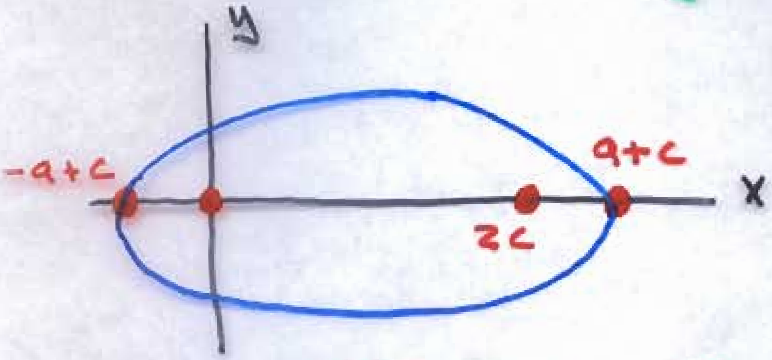
HITT Problem for Period 4

Consider an ellipse with

* semi-major axis a, along x axis
* semi-minor axis b, parallel to y
* one focus at $x=y=0$



* recall $\begin{Bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{Bmatrix} \Longleftrightarrow \begin{Bmatrix} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}(y/x) \end{Bmatrix}$

What is $r(\theta)$ on the ellipse?

(A) $r = a + c$

(B) $r = a + c\cos(\theta)$

(C) $r = \dfrac{b^2}{a - c\cos(\theta)}$

(D) $r = a + c\tan(\theta)$

(E) $r = a + c\sec(\theta)$

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HITT Problem for Period 5

Consider $\vec{x}(t) = r(t)\left[\cos(\theta(t))\hat{\imath} + \sin(\theta(t))\hat{\jmath}\right]$

for general $r(t)$ & $\theta(t)$.

What is $\dfrac{\vec{x} \cdot \vec{a}}{r}$?

(A) $2\dot{r}\dot{\theta} + r\ddot{\theta}$

(B) $\ddot{r}$

(C) $-r\dot{\theta}^2$

(D) $\ddot{r} - r\dot{\theta}^2$

(E) $r\ddot{\theta}$