Review - Nov. 30th 2004

Chapters: 10, 11, 12, 13, 15, 16

Review of rotational variables (scalar notation)

Angular position: $$\theta = \frac{s}{r} \quad (\text{in radians})$$

Angular displacement: $$\Delta\theta = \theta_2 - \theta_1$$

Average angular velocity: $$\omega_{avg} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous angular velocity: $$\omega = \operatorname*{Lim}_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Average angular acceleration: $$\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous angular acceleration: $$\alpha = \operatorname*{Lim}_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

Relationships between linear and angular variables

TABLE 11-1 Equations of Motion for Constant Linear Acceleration and for Constant Angular Acceleration

Equation Number	Linear Equation	Missing Variable		Angular Equation	Equation Number
(2-11)	$v = v_0 + at$	$x - x_0$	$\theta - \theta_0$	$\omega = \omega_0 + \alpha t$	(11-12)
(2-15)	$x - x_0 = v_0 t + \frac{1}{2}at^2$	v	ω	$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$	(11-13)
(2-16)	$v^2 = v_0^2 + 2a(x - x_0)$	t	t	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$	(11-14)
(2-17)	$x - x_0 = \frac{1}{2}(v_0 + v)t$	a	α	$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$	(11-15)
(2-18)	$x - x_0 = vt - \frac{1}{2}at^2$	v_0	ω_0	$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$	(11-16)

Position: $s = \theta r \quad (\theta \text{ in radians})$

Velocity: $$v = \frac{d\theta}{dt} r = \omega r$$

Time period for rotation: $$T = \frac{\text{circumference}}{\text{velocity}} = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$$

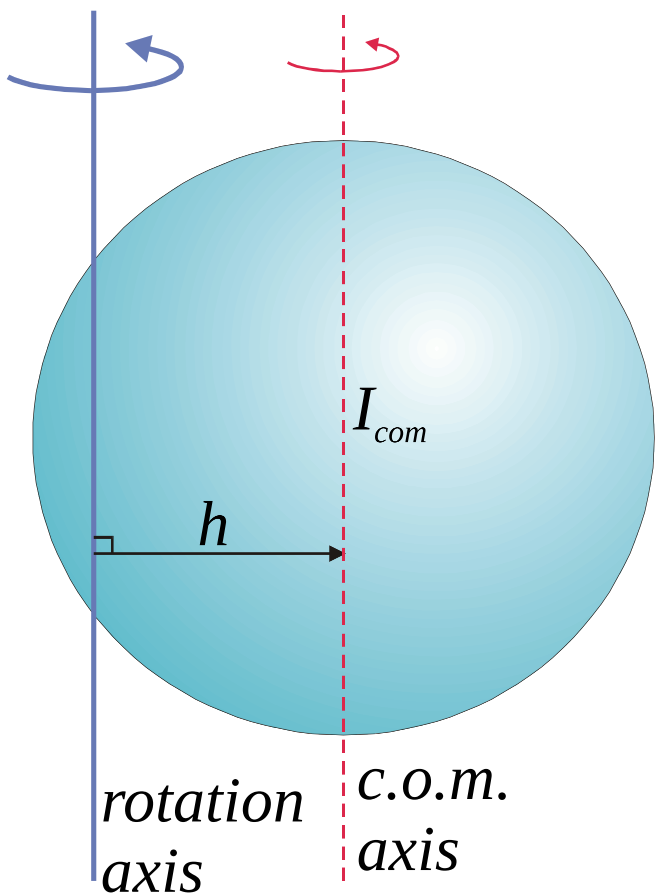
Tangential acceleration: $a_t = \alpha r$

Centripetal acceleration: $$a_r = \frac{v^2}{r} = \omega^2 r$$

Kinetic energy of rotation

$$K = \tfrac{1}{2} I \omega^2 \qquad I = \sum m_i r_i^2$$

Therefore, for a continuous rigid object: $I = \int r^2 dm = \int \rho r^2 dV$



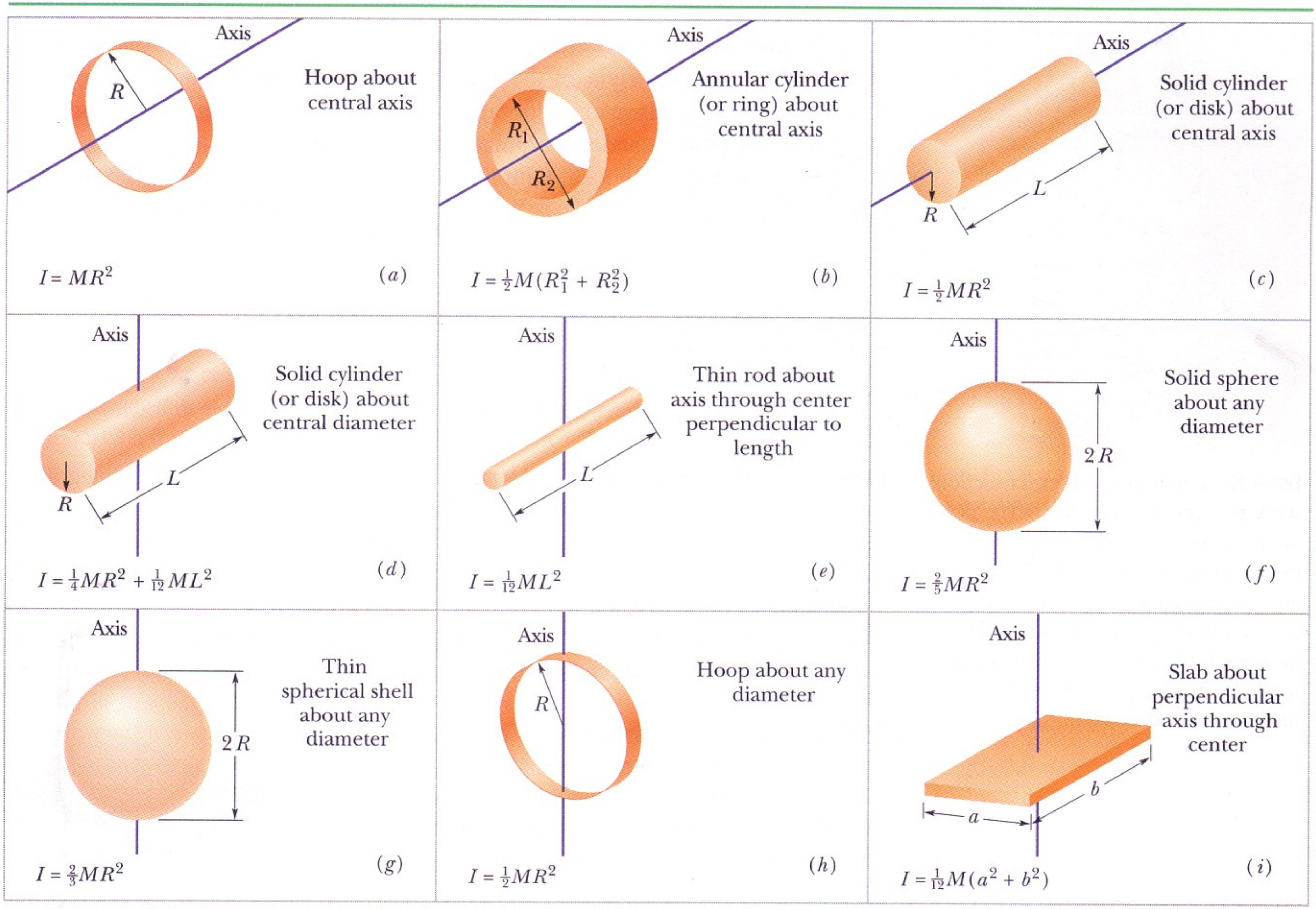
Parallel axis theorem

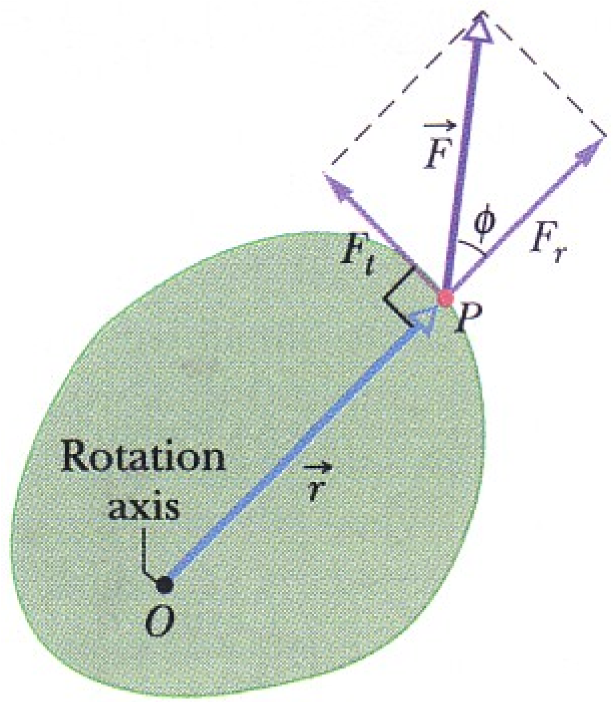
- If moment of inertia is known about an axis though the center of mass (*c.o.m.*), then the moment of inertia about any parallel axis is:

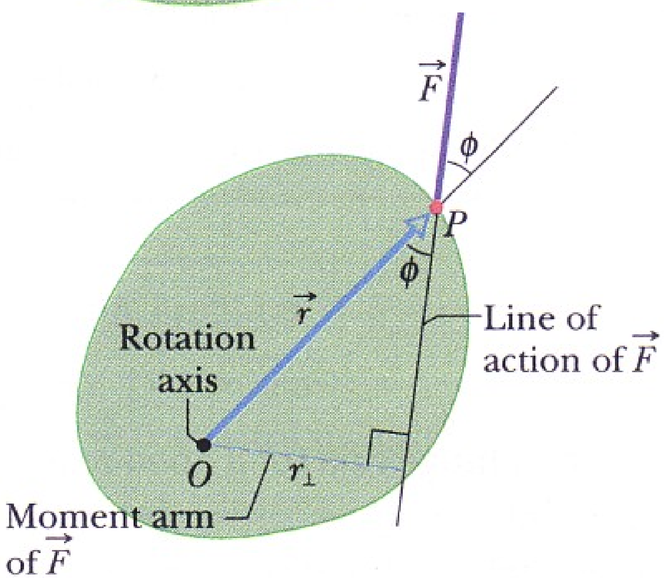
$$I = I_{com} + Mh^2$$

- It is essential that these axes are parallel; as you can see from table 10-2, the moments of inertia can be different for different axes.

Some rotational inertia







Torque

- There are two ways to compute torque:

$$\tau = (r)(F\sin\phi) = rF_t$$

$$= (r\sin\phi)(F) = r_\perp F$$

- The direction of the force vector is called the line of action, and $r_\perp$ is called the moment arm.

- The first equation shows that the torque is equivalently given by the component of force tangential to the line joining the axis and the point where the force acts.

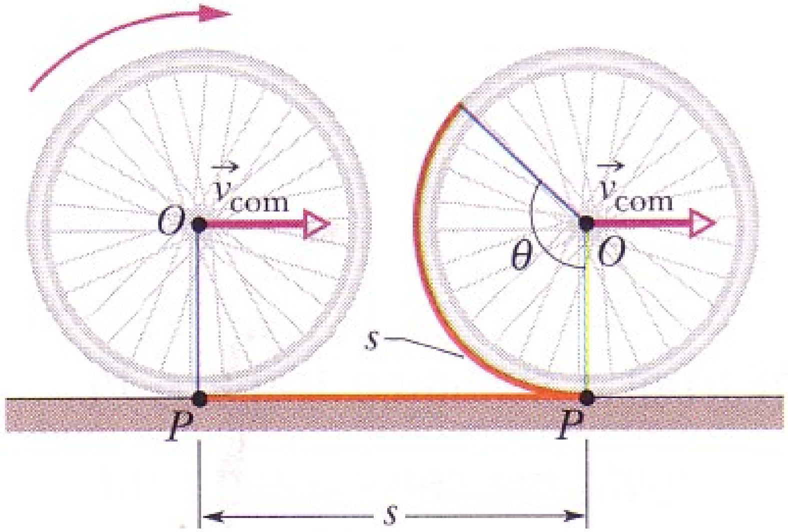
- In this case, r is the moment arm of F_t.

Summarizing relations for translational and rotational motion

Pure Translation (Fixed Direction)		Pure Rotation (Fixed Axis)	
Position	x	Angular position	θ
Velocity	$v = dx/dy$	Angular velocity	$\omega = d\theta/dt$
Acceleration	$a = dv/dt$	Angular acceleration	$\alpha = d\omega/dt$
Mass	m	Rotational inertia	I
Newton's second law	$F_{net} = ma$	Newton's second law	$\tau_{net} = I\alpha$
Work	$W = \int F\,dx$	Work	$W = \int \tau\,d\theta$
Kinetic energy	$K = \frac{1}{2}mv^2$	Kinetic energy	$K = \frac{1}{2}I\omega^2$
Power (constant force)	$P = Fv$	Power (constant torque)	$P = \tau\omega$
Work–kinetic energy theorem	$W = \Delta K$	Work–kinetic energy theorem	$W = \Delta K$

- Note: work obtained by multiplying torque by an angle - a dimensionless quantity. Thus, torque and work have the same dimensions, but you see that they are quite different.

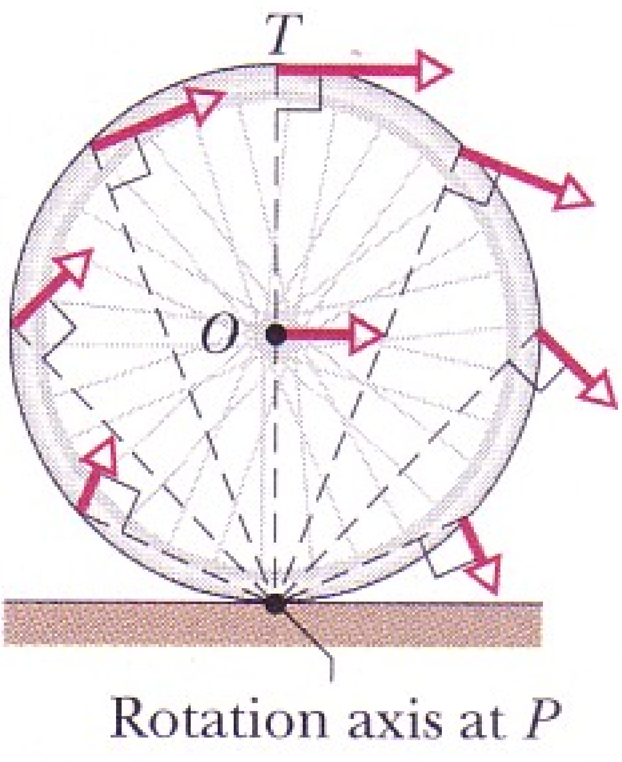
Rolling motion as rotation and translation



$$s = \theta R$$

The wheel moves with speed *ds/dt*

$$\Rightarrow v_{com} = \frac{d\theta}{dt} = \omega R$$



The kinetic energy of rolling

$$K = \tfrac{1}{2} I_P \omega^2 \qquad I_P = I_{com} + MR^2$$

$$K = \tfrac{1}{2} I_{com}\omega^2 + \tfrac{1}{2} MR^2\omega^2$$

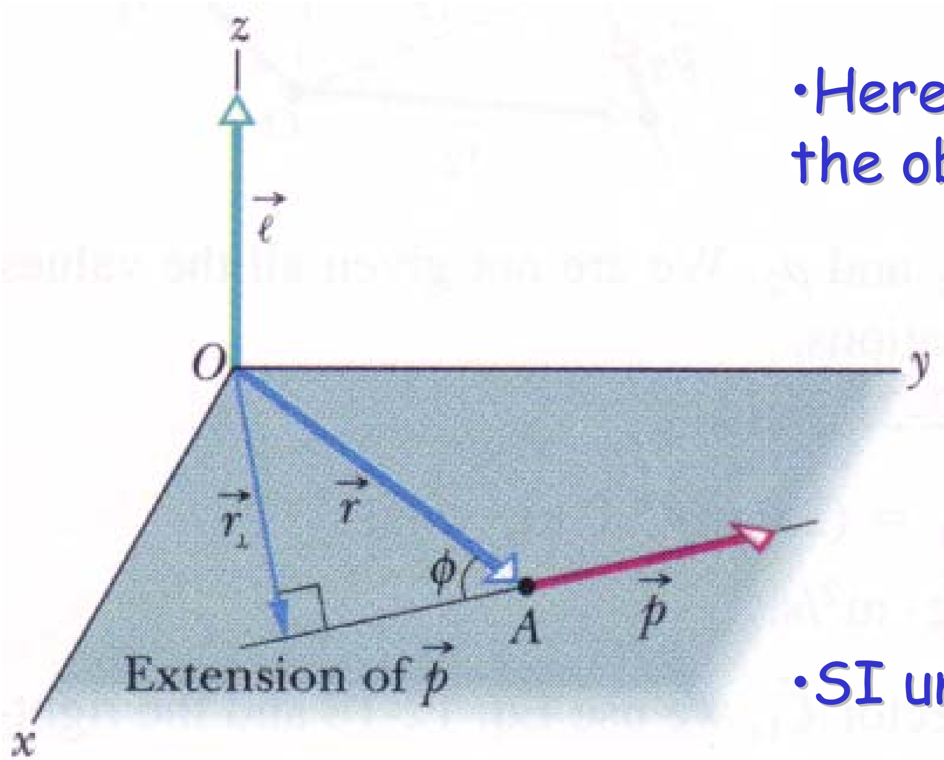
$$K = \tfrac{1}{2} I_{com}\omega^2 + \tfrac{1}{2} Mv_{com}^2 = K_r + K_t$$

Torque and angular momentum

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \text{(definition)}$$

• Torque was discussed in the previous chapter; cross products are discussed in chapter 3 (section 3-7) and at the end of this presentation; torque also discussed in this chapter (section 7).

Angular momentum $\vec{l}$ is defined as: $\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$



• Here, p is the linear momentum mv of the object.

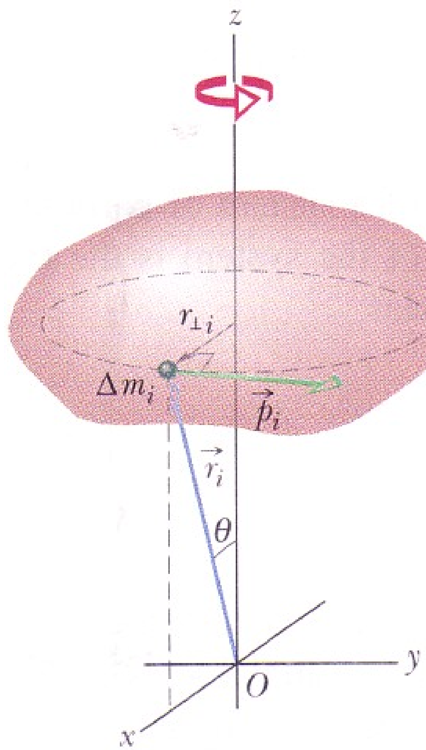
$$l = mvr\sin\phi$$

$$= rp_\perp = rmv_\perp$$

$$= r_\perp p = r_\perp mv$$

• SI unit is Kg.m^2/s.

Angular momentum of a rigid body about a fixed axis



We are interested in the component of angular momentum parallel to the axis of rotation:

$$L_z = \sum_{i=1}^{n} l_{iz} = \sum_{i=1}^{n} m_i v_i r_{\perp i} = \int v r_\perp dm$$

$$= \int (r_\perp \omega) r_\perp dm = \omega \int r_\perp^2 dm = I\omega$$

In fact: $\vec{L} = I\vec{\omega}$

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau}\ (= \vec{r} \times \vec{F})$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{\ell}\ (= \vec{r} \times \vec{p})$
Linear momentum[b]	$\vec{P}\ (= \Sigma \vec{p}_i)$	Angular momentum[b]	$\vec{L}\ (= \Sigma \vec{\ell}_i)$
Linear momentum[b]	$\vec{P} = M\vec{v}_{com}$	Angular momentum[c]	$L = I\omega$
Newton's second law[b]	$\vec{F}_{net} = \frac{d\vec{P}}{dt}$	Newton's second law[b]	$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$
Conservation law[d]	$\vec{P}$ = a constant	Conservation law[d]	$\vec{L}$ = a constant

Conservation of angular momentum

It follows from Newton's second law that:

If the net external torque acting on a system is zero, the angular momentum of the system remains constant, no matter what changes take place within the system.

$$\vec{L} = \text{a constant}$$

$$\vec{L}_i = \vec{L}_f$$

$$I_i\omega_i = I_f\omega_f$$

$$\frac{\omega_f}{\omega_i} = \frac{I_i}{I_f}$$

What happens to kinetic energy?

$$K_f = \tfrac{1}{2}I_f\omega_f^2 = \tfrac{1}{2}I_f\left(\frac{I_i^2\omega_i^2}{I_f^2}\right) = \frac{I_i}{I_f}\tfrac{1}{2}I_i\omega_i^2 = \frac{I_i}{I_f}K_i$$

- Thus, if you increase ω by reducing I, you end up increasing K.
- Therefore, you must be doing some work.
- This is a very unusual form of work that you do when you move mass radially in a rotating frame.
- The frame is accelerating, so Newton's laws do not hold in this frame

Equilibrium

A system of objects is said to be in equilibrium if:

1. The linear momentum $\vec{P}$ of its center of mass is constant.
2. Its angular momentum $\vec{L}$ about its center of mass, or about any other point, is also constant.

If, in addition, $\vec{L}$ and $\vec{P}$ are zero, the system is said to be in **static equilibrium**.

$$\vec{F}_{net} = \frac{d\vec{P}}{dt} \quad \Rightarrow \vec{F}_{net} = 0$$

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt} \quad \Rightarrow \vec{\tau}_{net} = 0$$

1. The vector sum of all the external forces that act on a body must be zero.
2. The vector sum of all the external torques that act on a body, **measured about any axis**, must also be zero.

The requirements of equilibrium

1. The vector sum of all the external forces that act on a body must be zero.
2. The vector sum of all the external torques that act on a body, **measured about any axis**, must also be zero.

Balance of forces	Balance of torques
$\vec{F}_{net,x} = 0$	$\vec{\tau}_{net,x} = 0$
$\vec{F}_{net,y} = 0$	$\vec{\tau}_{net,y} = 0$
$\vec{F}_{net,z} = 0$	$\vec{\tau}_{net,z} = 0$

One more requirement for static equilibrium:

3. The linear momentum $\vec{P}$ of the body must be zero.

Elasticity

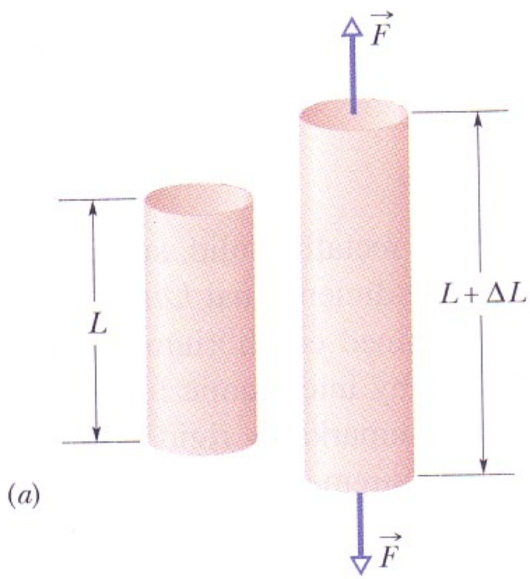
- All of these deformations have the following in common:
- A ***stress***, a force per unit area, produces a ***strain***, or dimensionless unit deformation.
- These various stresses and strains are related via a ***modulus of elasticity***

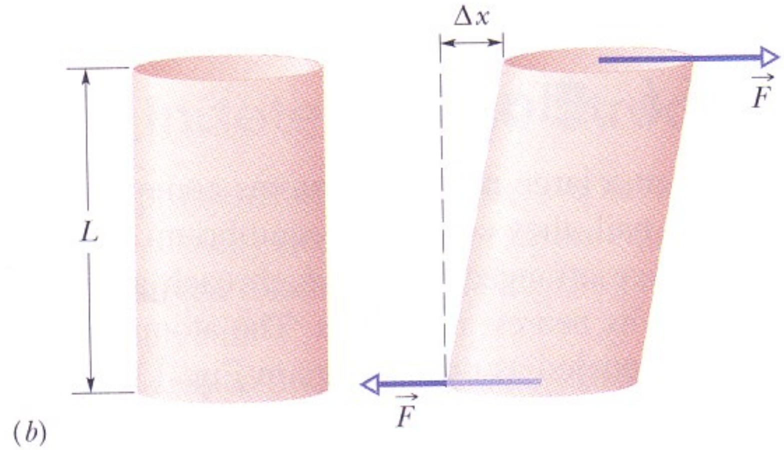
$$\text{stress} = \text{modulus} \times \text{strain}$$

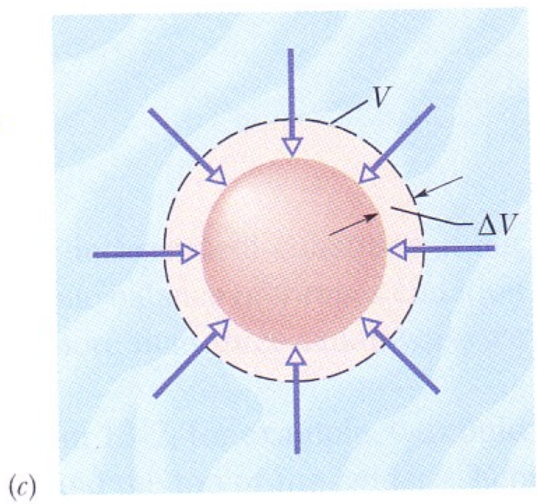
Tensile stress

Shear stress

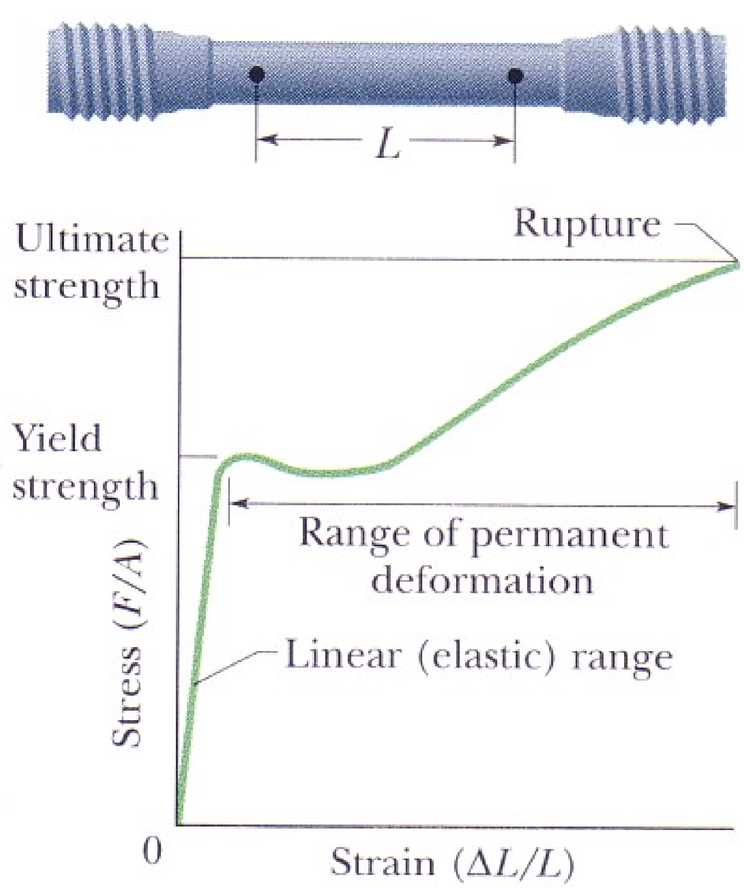
Hydraulic stress







Tension and compression

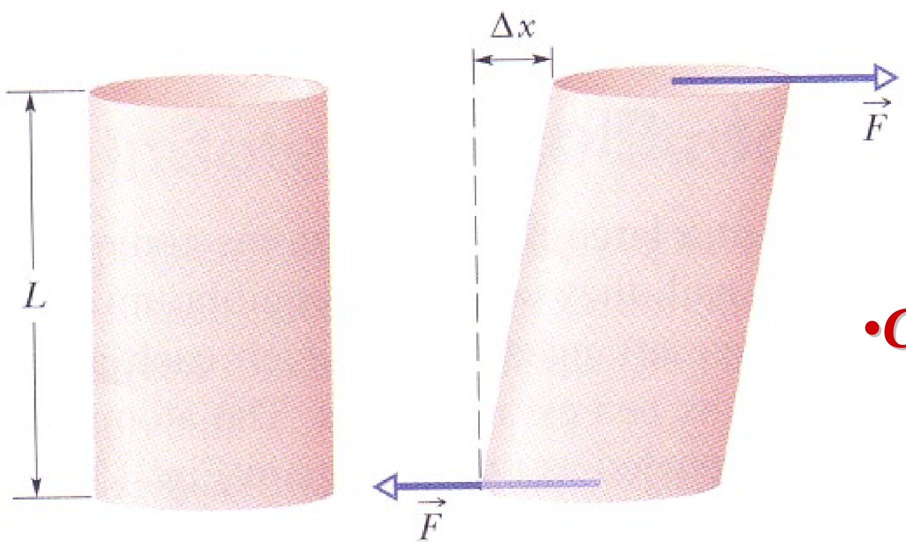


- The figure left shows a graph of stress versus strain for a steel specimen.
- Stress = force per unit area (*F*/ *A*)
- Strain = extension (Δ*L*) / length (*L*)
- For a substantial range of applied stress, the stress-strain relation is linear.
- Over this so-called ***elastic*** region, the, the specimen recovers its original dimensions when the stress is removed.
- In this region, we can write:

$$\frac{F}{A} = E\frac{\Delta L}{L}$$

This equation, **stress = *E* × strain**, is known as **Hooke's law**, and the modulus ***E*** is called **Young's modulus**. The dimensions of *E* are the same as stress, *i.e.* force per unit area.

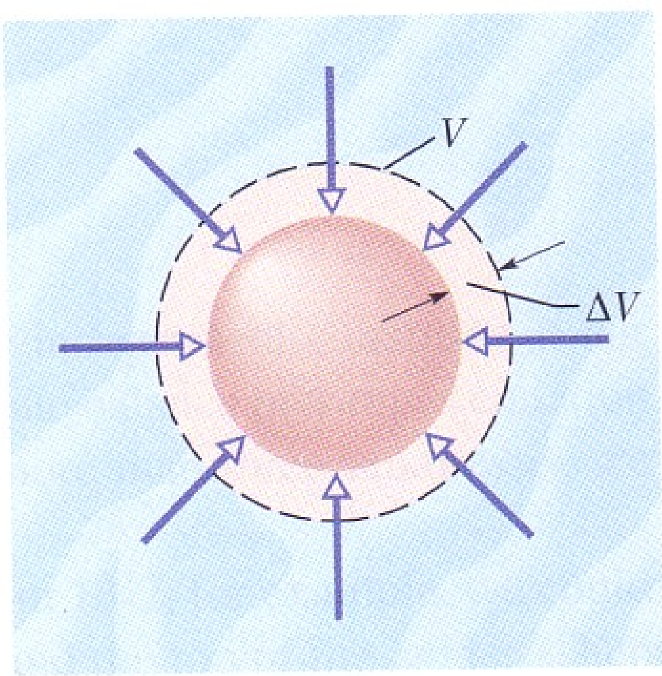
Shear stress



$$\frac{F}{A} = G\frac{\Delta x}{L}$$

- ***G*** is called the **shear modulus**.

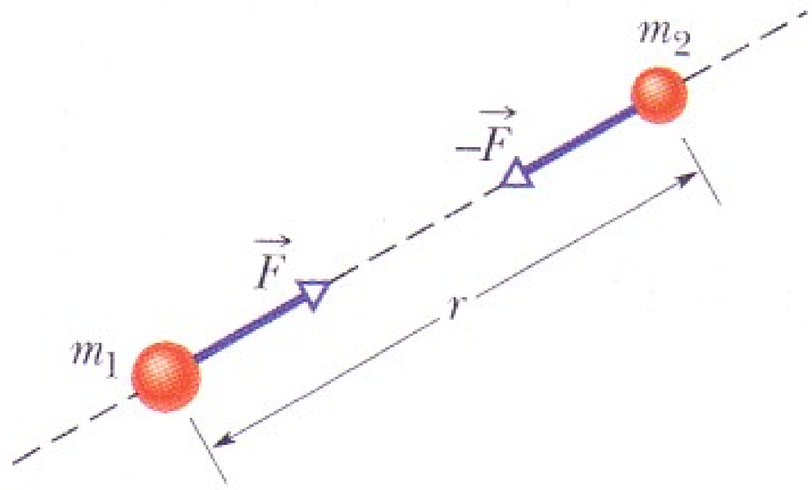
Hydraulic stress



$$p = B\frac{\Delta V}{V}$$

- ***B*** is called the **bulk modulus**.
- ***V*** is the volume of the specimen, and **ΔV** its change in volume under a hydrostatic pressure ***p***.

Newton's law of gravitation



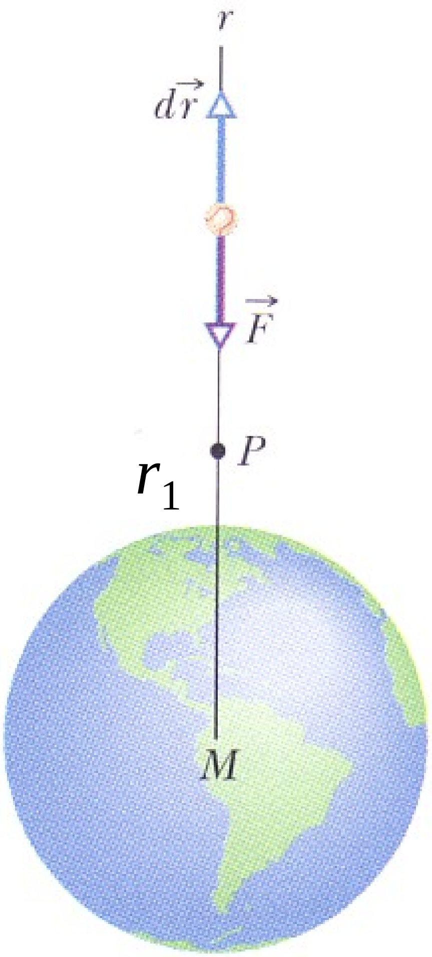
$$F = G\frac{m_1 m_2}{r^2} \qquad G = 6.67\times10^{-11}\,\mathrm{N\cdot m^2/kg^2}$$

Shell theorems

A uniform spherical shell of matter attracts a particle that is outside the shell as if the shell's mass were concentrated at its center.

A uniform spherical shell of matter exerts no net gravitational force on a particle located inside it

Gravitational potential energy



$$W_g = GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \frac{GM}{r_1 r_2} m(r_1 - r_2)$$

- But, close to the Earth's surface,

$$\frac{GM}{r_1 r_2} \approx \frac{GM}{\overline{r}^2} \approx g$$

$$i.e. \quad \Delta U = -W_g = mgh$$

- Further away from Earth, we must choose a reference point against which we measure potential energy.

The natural place to chose as a reference point is $r = \infty$, since U must be zero there, *i.e.* we set $r_1 = \infty$ as our reference point.

$$U = -W_g = \frac{GmM}{r_1} - \frac{GmM}{r_2} = -\frac{GmM}{r}$$

Planets and satellites: Kepler's laws

1. THE LAW OF ORBITS: All planets move in elliptical orbits, with the sun at one focus.

2. THE LAW OF AREAS: A line that connects a planet to the sun sweeps out equal areas in the plane of the planet's orbit in equal times; that is, the rate dA/dt at which it sweeps out area A is constant.

$$\frac{dA}{dt} \approx \tfrac{1}{2} r^2 \frac{d\theta}{dt} = \tfrac{1}{2} r^2 \omega$$

3. THE LAW OF PERIODS: The square of the period of any planet is proportional to the cube of the semimajor axis of the orbit.

$$T^2 = \frac{(2\pi)^2}{\omega^2} = \frac{(2\pi)^2}{GM} r^3$$

Satellites: Orbits and Energy

- Again, we'll do the math for a circular orbit, but it holds quite generally for all elliptical orbits.
- Applying *F = ma*:

$$\frac{GMm}{r^2} = (m)\left(\frac{v^2}{r}\right)$$

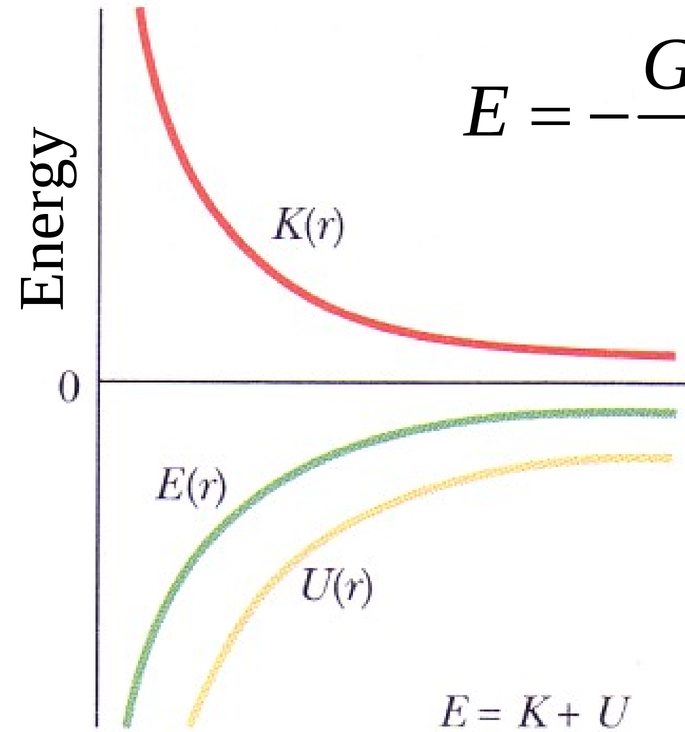
- Thus,

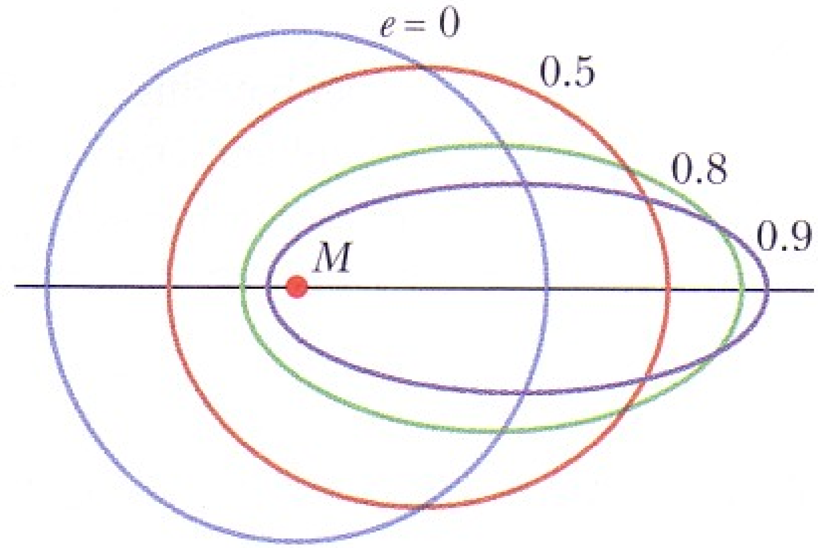
$$K = \tfrac{1}{2}mv^2 = \frac{GMm}{2r} = -\frac{U}{2}$$

$$E_{total} = K + U = \frac{GMm}{2r} - \frac{GMm}{r}$$

$$= -\frac{GMm}{2r} = -K$$

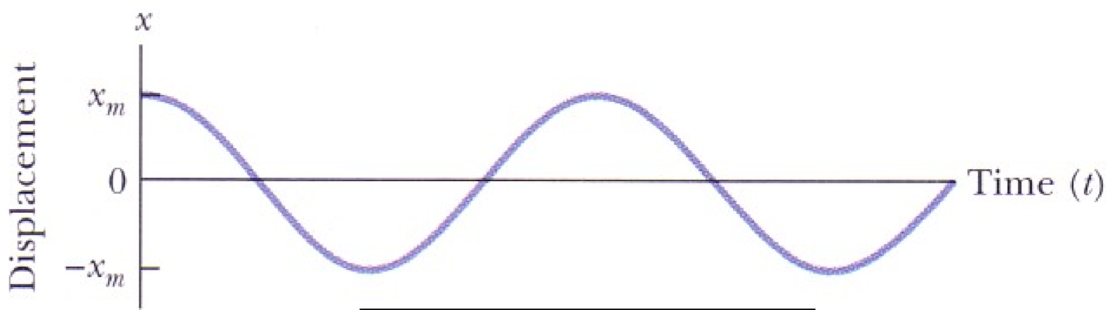
$$E = -\frac{GMm}{2a}$$



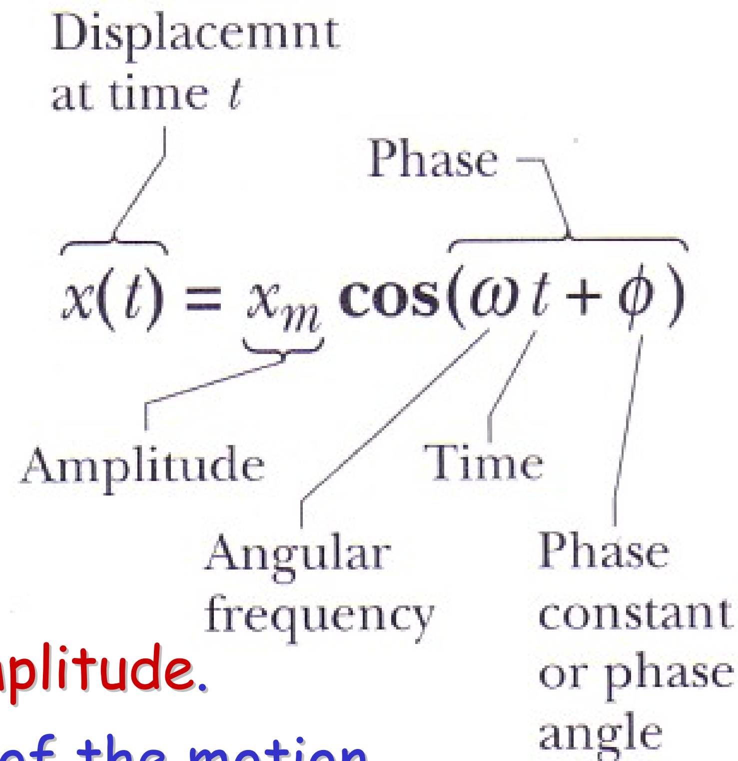


Simple Harmonic Motion

- The simplest possible version of harmonic motion is called Simple Harmonic Motion (SHM).
- This term implies that the periodic motion is a sinusoidal function of time,



$$\omega = \frac{2\pi}{T} = 2\pi f$$



- The positive constant x_m is called the amplitude.
- The quantity $(\omega t + \phi)$ is called the phase of the motion.
- The constant ϕ is called the phase constant or phase angle.
- The constant ω is called the angular frequency of the motion.
- T is the period of the oscillations, and f is the frequency.

The velocity and acceleration of SHM

Velocity:

$$v(t) = \frac{dx(t)}{dt} = \frac{d}{dt}\left[x_m \cos\left(\omega t + \phi\right)\right]$$

$$v(t) = -\omega x_m \sin\left(\omega t + \phi\right)$$

- The positive quantity ωx_m is called the velocity amplitude v_m.

Acceleration:

$$a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}\left[-\omega x_m \sin\left(\omega t + \phi\right)\right]$$

$$a(t) = -\omega^2 x_m \cos\left(\omega t + \phi\right)$$

$$a(t) = -\omega^2 x(t)$$

In SHM, the acceleration is proportional to the displacement but opposite in sign; the two quantities are related by the square of the angular frequency

The force law for SHM

$$F = ma = m(-\omega^2 x) = -(m\omega^2)x$$

•Note: SHM occurs in situations where the force is proportional to the displacement, and the proportionality constant $(-m\omega^2)$ is negative, *i.e.*

$$F = -kx$$

•This is very familiar - it is Hooke's law.

SHM is the motion executed by a particle of mass *m* subjected to a force that is proportional to the displacement of the particle but of opposite sign.

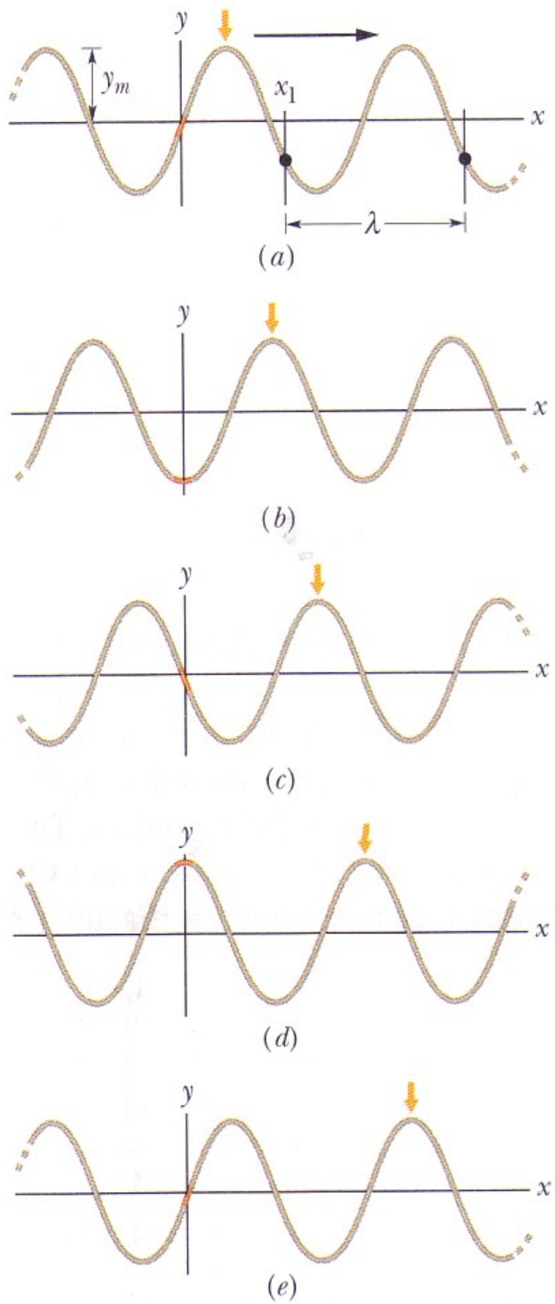
$$\omega = \sqrt{\frac{k}{m}} \qquad T = 2\pi\sqrt{\frac{m}{k}}$$

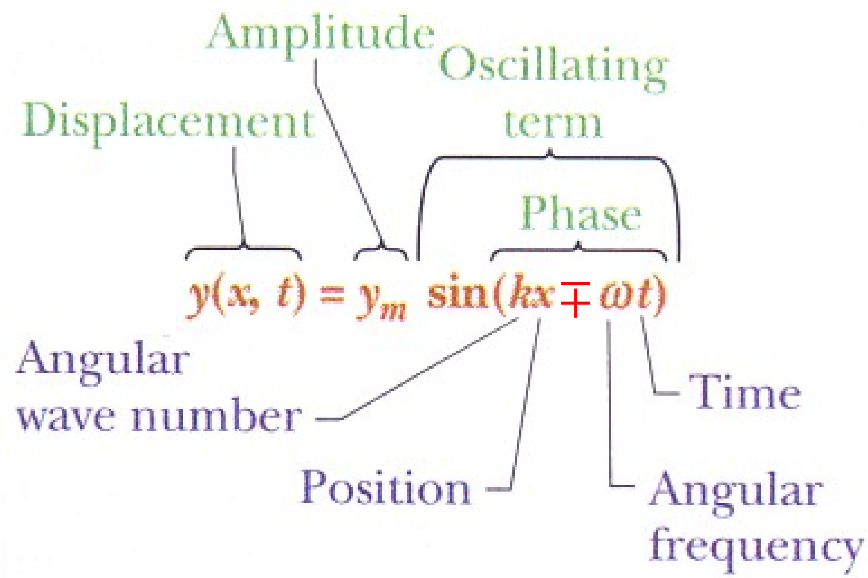
Mechanical energy: $E = U + K = \frac{1}{2}kx_m^2$

x_m is the maximum displacement or amplitude

Waves I - wavelength and frequency

Transverse wave





$$k = \frac{2\pi}{\lambda}$$ *k* is the angular wavenumber.

$$\omega = \frac{2\pi}{T}$$ ω is the angular frequency.

frequency $$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

velocity $$v = \pm\frac{\omega}{k} = \pm\frac{\lambda}{T} = \pm f\lambda$$

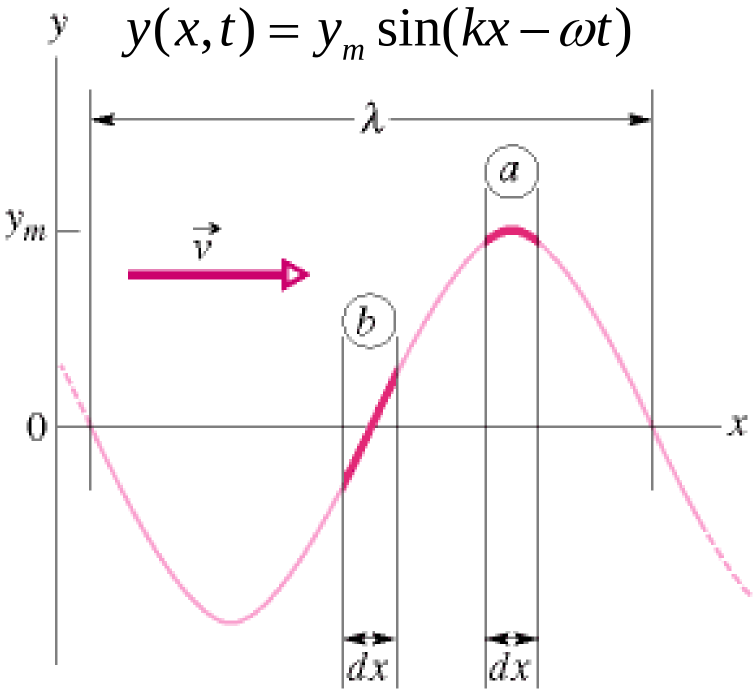
Review - traveling waves on a string

Velocity $$v = \sqrt{\frac{\tau}{\mu}}$$

- The tension in the string is τ.
- The mass of the element dm is μdl, where μ is the mass per unit length of the string.

Energy transfer rates

$$y(x,t) = y_m \sin(kx - \omega t)$$



$$P_{kinetic} = \frac{dK}{dt} = \tfrac{1}{2}\mu v \omega^2 y_m^2 \cos^2(kx - \omega t)$$

$$P_{elastic} = \frac{dU}{dt} = \tfrac{1}{2}\mu v \omega^2 y_m^2 \cos^2(kx - \omega t)$$

$$P_{avg} = 2 \times \tfrac{1}{2}\mu v \omega^2 y_m^2 \left\langle \cos^2(kx - \omega t) \right\rangle$$

$$= 2 \times \tfrac{1}{2}\mu v \omega^2 y_m^2 \times \tfrac{1}{2} = \tfrac{1}{2}\mu v \omega^2 y_m^2$$