Chapter 15: Oscillations

Lecture 35
11/23/2009
Oscillations

Goals for this Lecture:

- Displacement, velocity and acceleration of a simple harmonic oscillator
- Energy of a simple harmonic oscillator
- Examples of simple harmonic oscillators: spring-mass system, simple pendulum, physical pendulum, torsion pendulum
- Damped harmonic oscillator
- Forced oscillations/Resonance
Consider an object undergoing uniform circular motion:

- Uniform angular velocity \( \omega \Rightarrow \) the angle \( \theta \) increases linearly with time: \( \theta = \omega t + \Phi \)
- The \( x/y \) positions of the object is:
  \[ x = R \cos \theta, \; y = R \sin \theta. \] Then
  \[ x(t) = R \cos \theta = R \cos(\omega t + \Phi) \]
  \[ y(t) = R \sin \theta = R \sin(\omega t + \Phi) = R \cos(\omega t + \Phi - \pi/2) \]
- The velocity along the \( x/y \)-axis:
  \[ v_x = -\omega R \sin(\omega t + \Phi), \; v_y = -\omega R \sin(\omega t + \Phi - \pi/2) \]
- The acceleration along the \( x/y \)-axis:
  \[ a_x = -\omega^2 R \cos(\omega t + \Phi), \; a_y = -\omega^2 R \cos(\omega t + \Phi - \pi/2) \]
Damped Simple Harmonic Motion

So far we have considered oscillation without friction. If drag or friction act on the oscillator the resulting motion is damped.

Consider the forces acting on m

- Spring: \( F_s = -kx \)
- Drag: \( F_d = -bv \) (recall from start of semester)
- \( F_{net} = ma = -kx - bv \)
  \[ \Rightarrow ma + bv + kx = 0 \]
  \[ \Rightarrow m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0 \]
Damped Simple Harmonic Motion

Consider the forces acting on \( m \)

\[
F_{\text{net}} = ma = -kx - bv
\]

\[
\Rightarrow ma + bv + kx = 0
\]

\[
\Rightarrow m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0
\]

\[
\Rightarrow x(t) = x_m e^{-bt/2m} \cos(\omega' t + \Phi)
\]

\[
\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}
\]

exponentially decaying term
Energy of the Damped Oscillator

- Damping results in:
  - Decreasing amplitude
  - Smaller angular frequency: $\omega \Rightarrow \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$
  - Longer period
  - The energy is decreasing exponentially
    $E(t) = \frac{1}{2} k x_m^2 \Rightarrow E(t) = \frac{1}{2} k x_m^2 e^{-bt/m}$
    (the system exponentially loses energy)
Forced/Driven Oscillations

- **Free oscillation:**
  - after an initial "push" the system is left on its own.
  - The body oscillates with its "natural" frequency, \( \omega \).

- **Forced (driven) oscillation:**
  - a periodic force is continuously applied to the oscillating body:
    \[
    F(t) = F_m \cos(\omega_d t + \Phi)
    \]
  - The body oscillates with the frequency of the driving force, \( \omega_d \).
Forced/Driven Oscillations

- Forced (driven) oscillation:
  - a periodic force is continuously applied to the oscillating body: $F(t) = F_m \cos(\omega_d t + \Phi)$
  - The body oscillates with the frequency of the driving force, $\omega_d$: $x(t) = x_m \cos(\omega_d t + \Phi)$
  - The oscillation amplitude $x_m$ depends on the driving frequency $\omega_d$
Forced Oscillations & Resonance

- **Forced (driven) oscillation:**
  - $F(t) = F_m \cos(\omega_d t + \Phi)$
  - $x(t) = x_m \cos(\omega_d t + \Phi)$
  - The oscillation amplitude $x_m$ depends on the driving frequency $\omega_d$. $x_m$ is largest when $\omega = \omega_d$. This condition is called resonance

http://www.youtube.com/watch?v=HxTZ446tbzE
http://www.youtube.com/watch?v=xlOS_31Ubdo
http://www.youtube.com/watch?v=j-zczJXSxnw&NR=1&feature=fvwp
Example

The suspension system of a 2000 kg automobile “sags” 10 cm when the chassis is placed on it. Also, the oscillation amplitude decreases by 50% each cycle. Assuming each wheel supports 500 kg.

a) Estimate the values of the spring constant $k$

b) Estimate the values of the damping constant $b$ for the spring and shock absorber system of one wheel
Example

a) Estimate the values of the spring constant $k$

$$F_{\text{max}} = k \ x_{\text{max}} \Rightarrow mg = k \ x_{\text{max}} \Rightarrow k = \frac{mg}{x_{\text{max}}}$$

$$k = \frac{(500 \ \text{kg})(9.81 \ \text{m/s}^2)}{(0.1 \ \text{m})} = 49 \times 10^3 \ \text{N/m}$$

b) Estimate the values of the damping constant $b$ for the spring and shock absorber system of one wheel:

$$x(t) = x_m \ e^{-bt/2m} \cos(\omega' t + \Phi)$$

decreases 50% each cycle

$$\Rightarrow e^{-bT/2m} = 0.5$$

where $T = \frac{2\pi}{\omega'} \approx \frac{2\pi}{\omega} = \frac{2\pi \sqrt{m/k}}{}$

$$\Rightarrow e^{-\pi b \sqrt{(1/mk)}} = 0.5 \Rightarrow \pi b \sqrt{(1/mk)} = 0.693$$

$$b = 1092 \ \text{kg/s}$$