2048_Fall13_Exam2: solutions to 2nd exam.

1. A particle starts from rest at time t = 0 and moves along the x axis. If the net force on it is proportional to the time t, its kinetic energy is proportional to: (ans = t^4). Re-express kinetic energy in terms of momentum using $p \equiv mv$ to say $K = \frac{1}{2}mv^2 = \frac{1}{2m}p^2$; then, we can say $K = \frac{1}{2m}p^2 \xleftrightarrow{\text{solve for p}} p = \sqrt{2mK}$. Meanwhile, the statement that force is proportional to t means F is a constant multiple of t. Call this constant multiple[1] "j".

$$jt = F = \frac{dp}{dt} = \frac{d}{dt}\sqrt{2mK} \leftrightarrow d\sqrt{2mK} = jt \cdot dt \leftrightarrow \int d\sqrt{2mK} = \int jt \cdot dt \leftrightarrow \sqrt{2mK} = j\frac{1}{2}t^2; \tag{1.1}$$

Solving for K in (1.1),

$$K = \frac{1}{2m}\left(j\tfrac{1}{2}t^2\right)^2 \propto \boxed{t^4} \tag{1.2}$$

2. An object attached to a cord is moving in a circular path of radius 0.5m on a horizontal frictionless surface. The cord will break if its tension exceeds 16 N. The maximum kinetic energy the object can have is: (ANS = 4J).

A very simple calculation requiring $K = \frac{1}{2}mv^2$ and $a_r = v^2 / r = F_r / m$,

$$K = \frac{1}{2}mv^2 \times \frac{r}{r} = \frac{1}{2}r \times m\frac{v^2}{r} = \frac{1}{2}r \times F_r = \frac{1}{2}(0.5m)\times(16N) = 4N \cdot m = \boxed{4J}; \tag{1.3}$$

3. A 0.50-kg object moves in a horizontal circular track with a radius of 2.5 m. An external force of 3.0 N, always tangent to the track, causes the object to speed up as it goes around. The work done by the external force as the mass makes one revolution is (ANS = 47 J).

"Always tangent to the track" means the angle between the force and the displacement is 0 degrees,

$$W = \mathbf{F} \bullet \mathbf{d} = |F||d|\cos\theta_{\mathbf{Fd}} = |F||2\pi r|\cos 0° = |3.0N||2\pi \cdot 2.5m|\cos 0° = \boxed{15\pi\ N \cdot m \approx 47\ \mathrm{N \cdot m}}; \tag{1.4}$$

4. The potential energy of a body of mass m is given by $U = -mgx + kx^2/2$. The corresponding force is (ANS = mg-kx). The equation $F_x = -\frac{dU}{dx}$ is absolutely fundamental[2]; it pops up in the formalism of advanced physics,

$$F_x = -\frac{dU}{dx} = -\frac{d}{dx}\left(-mgx + \tfrac{1}{2}kx^2\right) = -\left(-mg + \tfrac{1}{2}k2x\right) = \boxed{mg - kx}; \tag{1.5}$$

*5. A 4-kg particle moves along the x axis under the influence of a conservative force. The potential energy is given by U (x) = (1.0 J/m^3)*x^3 where x is in meters. If the velocity of the particle at x = −2 m is 2 m/s, how far to the right along x (x ≥ 0) does the particle reach? (ANS = 0 m)*

The kinetic energy at x = -2m is $K = \frac{1}{2}mv^2 = \frac{1}{2}(4kg)(2\frac{m}{s})^2 = 8J$. This velocity is in the +x direction. Meanwhile, U(x) appears as a cubic potential. At x = 2, the potential energy of the particle is $U(-2m) = 1.0\frac{J}{m^3}(-2.0m)^3 = -8.0J$. The total energy of the system is,

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[1] Notice that the units of j are F/t = ma/t, so we could think of "j" as "jerk".

[2] Its vector-generalization is: $\mathbf{F} = -\hat{x}\frac{dU}{dx} - \hat{y}\frac{dU}{dy} - \hat{z}\frac{dU}{dz} \equiv -\nabla U$.

$$E = E(x,v) = K(v) + U(x) = \frac{1}{2}mv^2 + 1.0\frac{J}{m^3}x^3 \to E(-2m, 2\tfrac{m}{s}) = 8 - 8J = 0J;$$
$$E(x^*, 0) = 0 = 1.0\frac{J}{m^3}x^{*3} \leftrightarrow \boxed{x^* = \sqrt[3]{0} = 0m}; \tag{1.6}$$

6. A stone of mass M rests on an elastic spring which is compressed a distance of 1 cm by the weight of the stone. The stone is pushed down an additional distance of 3 cm and then released. What is the maximum height reached by the stone relative to the release point? (ANS = 8 cm) Let the stone have 0 potential energy when it does not compress the spring at all. Let $x_0 = 1cm$, $x' = 4cm$, and $\Delta x = 3cm$. Let down be the negative direction. Then, gravitational potential energy is $U_g = Mgx$, and spring-poential-energy is $U_s = -\frac{1}{2}kx^2$ (using (1.5), we check: $F_S = -\frac{dU_S}{dx} = --kx = kx$, and since $x > 0$, the spring-force points in the +x direction, as it should). In all of these, we obviously have $0 \le x \le x'$. We also have a spring constant $k = \Delta F / \Delta x = Mg / x_0$ Then, the potential energy when fully-compressed is given by,

$$U = \frac{1}{2}kx^2 - Mgx = U(x) = \frac{1}{2}\frac{Mg}{x_0}x^2 - Mgx = Mgx\left(\frac{1}{2}\frac{x}{x_0} - 1\right); \quad U(0) = \frac{1}{2}k0^2 - Mg \cdot 0 = 0; \tag{1.7}$$

We have $U(0) = 0$, meaning our potential energy (1.7) satisfies what we initially said about the stone having 0 potential enegy when the spring is not compressed at all.

So: $U_i = U(x') = Mg(4cm)(\frac{1}{2}\frac{4cm}{1cm} - 1) = 4Mg \cdot cm$, and $U_0 = U(x_0) = Mg(1cm)(\frac{1}{2}\frac{1cm}{1cm} - 1) = -\frac{1}{2}Mg \cdot cm$. The kinetic energy this particle will have is $K = U_i + U_0 = \frac{7}{2}Mg \cdot cm$. The stone will then rise by $\Delta y = \frac{K}{Mg} = \frac{7}{2}cm$. Adding the 4 cm of initial compression to this, we get 7.5 cm, or 8 cm on truncating to 1 significant figure. (in retrospect, we could have prentended the spring wasn't initially compressed to 1 cm).

7. With a help of a winch, one attempts to raise a 6-ton block vertically up by 1 m by sliding it along a frictionless incline that makes an angle of 20 degrees with the horizon. If the task is to be accomplished in 5 min, what is the minimal power requirement for the winch?

$$P = \frac{\Delta W}{\Delta t} = \frac{\Delta(mgh)}{\Delta t} = mg\frac{\Delta h}{\Delta t} = (6000kg)\left(10\tfrac{m}{s^2}\right)\frac{1m}{(300\text{ s})} = \boxed{200W}; \tag{1.8}$$

8. The figure shows a cubical box with each side consisting of a uniform metal plate of negligible thickness. Each of the four sides have mass, M, and the bottom has mass 2M. The box is open at the top (at z = L) and has edge length L. What is the z-coordinate of the center-of-mass? Caution: the bottom plate has mass $2M$,

$$z_{CM} = \frac{\sum m_i z_i}{\sum m_i} = \frac{\sum m_i z_i^{CM}}{\sum m_i} = \frac{M \cdot \frac{L}{2} + M \cdot \frac{L}{2} + M \cdot \frac{L}{2} + M \cdot \frac{L}{2} + M \cdot 0}{M + M + M + M + 2M} = \frac{2ML}{6M} = \boxed{\frac{L}{3}} \tag{1.9}$$

9. A 75-kg man is riding in a 30-kg cart at 2.0 m/s. He jumps off in such a way as to land on the ground with no horizontal velocity. The resulting change in speed of the cart is,

$$p = p' \to mv = m'v' \to \Delta v = v' - v = \frac{m}{m'}v - v = \left(\frac{m}{m'} - 1\right)v = \left(\frac{(75+30)kg}{30kg} - 1\right)\left(2.0\tfrac{m}{s}\right) = \boxed{5\frac{m}{s}} \tag{1.10}$$

10. A firefighter puts down fire. He holds a hose ejecting 400 liters of water per minute through a nozzle of 1 cm in diameter. What is the force exerted by the hose on the firefighter? The water density is 1 g/cm3; 1 liter is 10^{-3} m^3.

$$\begin{aligned} F &= \frac{\Delta p}{\Delta t} = \frac{\Delta(mv)}{\Delta t} = \frac{\Delta(\rho \boldsymbol{\upsilon} v)}{\Delta t} = \frac{\Delta \boldsymbol{\upsilon}}{\Delta t}\rho \cdot v = \frac{\Delta \boldsymbol{\upsilon}}{\Delta t}\rho \cdot \frac{\Delta L}{\Delta t} = \frac{\Delta \boldsymbol{\upsilon}}{\Delta t}\rho \cdot \frac{\Delta \boldsymbol{\upsilon} / A}{\Delta t} \\ &= \left(\frac{\Delta \boldsymbol{\upsilon}}{\Delta t}\right)^2 \frac{\rho}{\pi(\frac{D}{2})^2} = \left(\frac{0.4m^3}{60s}\right)^2 \frac{10^3 \frac{kg}{m^3}}{\pi(\frac{0.01m}{2})^2} = 566N = \boxed{600N} \end{aligned} \tag{1.11}$$

11. An astronaut is being tested in a centrifuge. The centrifuge has a radius R and, when starting from rest at t = 0, rotates according to θ(t) = (0.125rad/s^2)t^2 = (1/8 rad/s^2)t^2. At what time t > 0 is the magnitude of the tangential acceleration equal to the magnitude of the radial acceleration?

$$\begin{aligned} &a_r = a \rightarrow \frac{v^2}{r} = r\alpha \rightarrow \frac{(r\omega)^2}{r} = r\frac{d^2\theta}{dt^2} \rightarrow \frac{r^2}{r}\left(\frac{d\theta}{dt}\right)^2 = r\frac{d^2\theta}{dt^2} \\ &\leftrightarrow \left(\frac{d\theta}{dt}\right)^2 = \frac{d^2\theta}{dt^2} \rightarrow \left(\frac{2}{8}t^{2-1}\right)^2 = \left(\frac{2\cdot 1}{8}t^{2-2}\right) \leftrightarrow \boxed{t = 2\frac{rad}{s}} \end{aligned} \tag{1.12}$$

12. The rotational inertia of a solid uniform sphere about a diameter is (2/5)MR^2, where M is its mass and R is its radius. If the sphere is pivoted about an axis that is tangent to its surface, its rotational inertia is:

$$I_{\|} = I_0 + Mh^2 = \tfrac{2}{5}MR^2 + M(R)^2 = \left(\tfrac{2}{5}+\tfrac{5}{5}\right)MR^2 = \boxed{\tfrac{7}{5}MR^2}; \tag{1.13}$$

13. A rod is pivoted about its center. A 5-N force is applied 4 m from the pivot and another 5-N force is applied 2 m from the pivot, as shown. The magnitude of the total torque about the pivot (in N·m) is:

$$\sum \tau = r_1F_1\sin\theta_1 + r_2F_2\sin\theta_2 = (5N)(4m)\sin 30° + (5N)(2m)\sin 30° = (10+5)N\cdot m = \boxed{15\ N\cdot m}; \tag{1.14}$$

14. A 16-kg block is attached to a cord that is wrapped around the rim of a flywheel of diameter 0.40 m and hangs vertically, as shown. The rotational inertia of the flywheel is 0.50 kg·m^2. When the block is released and the cord unwinds, the acceleration of the block is:

$$\begin{aligned} &a = \frac{\sum F}{m} = \frac{mg - T}{m} = \frac{mg - \frac{1}{r}\sum\tau}{m} = \frac{mg - \frac{1}{r}I\alpha}{m} = \frac{mg - \frac{1}{r}I\frac{1}{r}a}{m} = g - \frac{I}{mr^2}a \\ &\xleftrightarrow{\text{solve for a}} a = \frac{g}{1+\frac{I}{mr^2}} = \frac{9.81\frac{m}{s^2}}{1+\frac{0.50kg\cdot m^2}{(16kg)(\frac{0.40}{2})^2}} = \boxed{5.51\frac{m}{s^2}}; \end{aligned} \tag{1.15}$$

15. A thin-walled hollow tube rolls without sliding along the floor. The ratio of its translational kinetic energy to its rotational kinetic energy (about an axis through its center of mass) is:

$$\frac{K_{trans}}{K_{rot}} = \frac{\frac{1}{2}mv^2}{\frac{1}{2}I_{hoop}\omega^2} = \frac{\frac{1}{2}mv^2}{\frac{1}{2}(mr^2)\omega^2} = \frac{v^2}{r^2\omega^2} = \frac{v^2}{v^2} = \boxed{1}; \tag{1.16}$$

16. A constant horizontal force Fapp of magnitude 20 N is applied to a wheel of mass 20 kg and radius 0.5 m as shown in the figure. The wheel rolls without slipping on the horizontal surface, and the acceleration of its center of mass has magnitude 0.5 m/s^2. What is the rotational inertia of the wheel about the rotation axis through its center of mass?

$$F_{app} = F_{CM} + F_{rot} = ma + \frac{\tau}{R\sin 90^\circ} = ma + \frac{I\alpha}{R} = ma + \frac{Ia/R}{R}$$

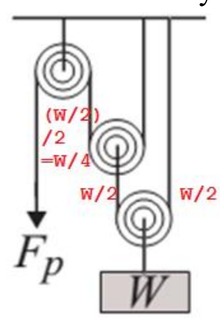
$$\xleftrightarrow{\text{solve for I}} I = R^2\left(\frac{F_{app}}{a} - m\right) = (0.5m)^2\left(\frac{20N}{0.5\frac{m}{s^2}} - 20kg\right) = \boxed{5\ kg\cdot m^2}; \qquad (1.17)$$

17. A mouse of mass M lies on the rim of a uniform disk of mass 4M that can rotate freely about its center like a merry-go-round. Initially the mouse and disk rotate together with an angular velocity of ω. If the mouse walks to a new position that is halfway to the center of the disk what is the new angular velocity of the mouse-disk system?

$$L_i = L_f \to \omega_f = \frac{I_i}{I_f}\omega_i = \frac{\frac{1}{2}(4M)r^2 + Mr^2}{\frac{1}{2}(4M)r^2 + M(\frac{1}{2}r)^2}\omega_i = \frac{\frac{1}{2}(4)+1}{\frac{1}{2}(4)+(\frac{1}{2})^2}\omega_i = \frac{3}{2.25}\omega_i = \boxed{\frac{4}{3}\omega_i}; \qquad (1.18)$$

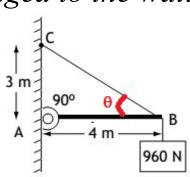
18. The ideal mechanical advantage is defined to be the ratio of the weight W tothe force of the pull FP for equilibrium (i.e., W/FP in equilibrium). Assuming that the pulleys are massless and there is no friction in the system, what is the ideal mechanical advantage of the combination of pulleys shown in the figure?

By noting the force to split in half at every pulley, we have,



$$\to F_P = \frac{1}{2^2}W = \frac{1}{4}W \to \frac{W}{F_P} = \frac{W}{\frac{1}{4}W} = \frac{1}{\frac{1}{4}} = \boxed{4}; \qquad (1.19)$$

19. A 960-N block is suspended on a rope attached to point B as shown. The beam AB is weightless and is hinged to the wall at A. The tension force of the cable attached to points C and B has magnitude:



$$\to T = \sqrt{T_x{}^2 + T_y{}^2} = \sqrt{(T_y\cot\theta)^2 + T_y{}^2} = T_y\sqrt{(\cot\tan^{-1}\tfrac{3}{4})^2 + 1^2} = (960N)(1.667...) = \boxed{1600N} \qquad (1.20)$$

20. A certain wire stretches 0.90 cm when outward forces with magnitude F are applied to each end. The same forces are applied to a wire of the same material but with three times the diameter and three times the length. The second wire stretches:

$$\left.\begin{array}{l}\frac{1}{\ell}\Delta\ell = \varepsilon = Y\sigma = Y(F/A) \\ \frac{1}{\ell'}\Delta\ell' = \varepsilon' = Y'\sigma' = Y'(F/A)\end{array}\right]; \quad \frac{\Delta\ell'}{\Delta\ell} = \frac{\ell' Y' \frac{F'}{A'}}{\ell Y\frac{F}{A}} = \frac{3\ell Y\frac{F}{3^2 A}}{\ell Y\frac{F}{A}} = \frac{3\frac{1}{3^2}}{1} = \frac{3}{3^2} = \frac{1}{3} \to \Delta\ell' = \Delta\ell\times\frac{1}{3} = \boxed{0.30cm} \qquad (1.21)$$

See? I told you that you only need TWO pieces of scrap paper—this would print out to 2 sheets front and back.