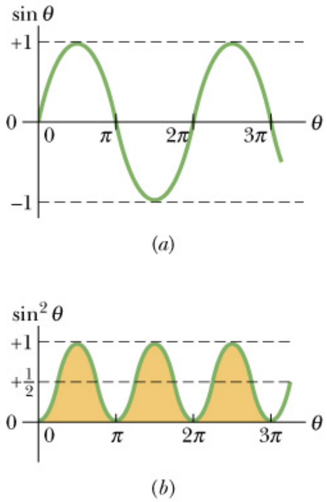
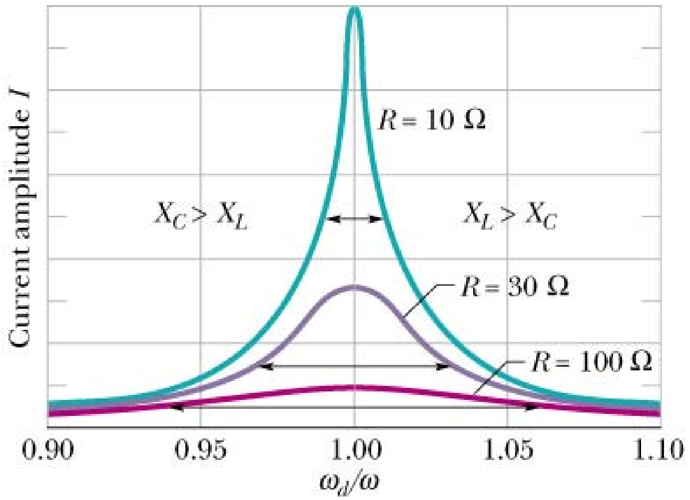
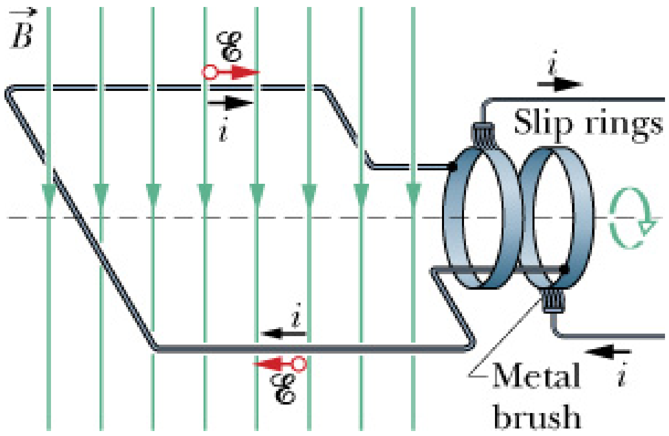
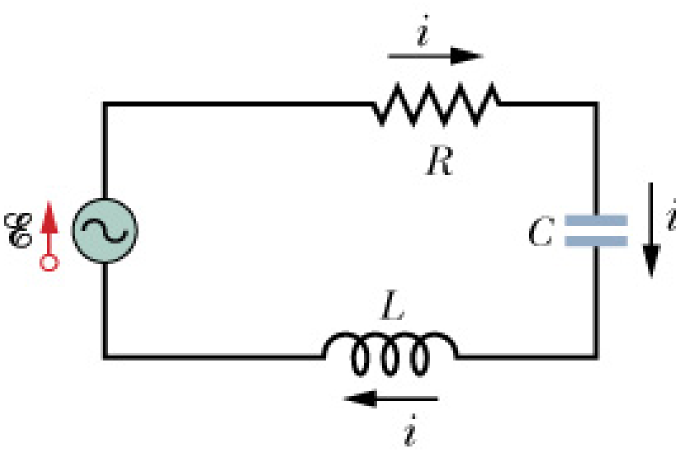
# Chapter 31: RLC Circuits



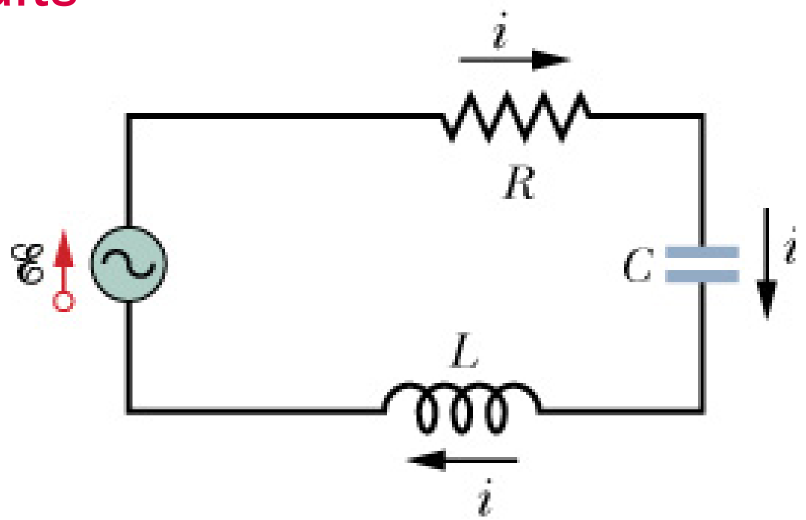



# Topics

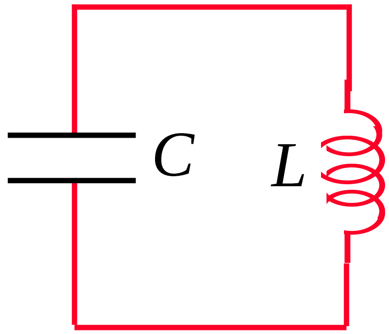
- ➔LC Oscillations
  - ◆ Conservation of energy
- ➔Damped oscillations in RLC circuits
  - ◆ Energy loss
- ➔AC current
  - ◆ RMS quantities
- ➔Forced oscillations
  - ◆ Resistance, reactance, impedance
  - ◆ Phase shift
  - ◆ Resonant frequency
  - ◆ Power
- ➔Transformers
  - ◆ Impedance matching



# LC Oscillations

➔Work out equation for LC circuit (loop rule)

$$-\frac{q}{C} - L\frac{di}{dt} = 0$$




➔Rewrite using i = dq/dt

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0 \Rightarrow \frac{d^2q}{dt^2} + \omega^2 q = 0$$

$$\omega = \frac{1}{\sqrt{LC}}$$

- $\omega$ (angular frequency) has dimensions of 1/t

➔Identical to equation of mass on spring

$$m\frac{d^2x}{dt^2} + kx = 0 \Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$\omega = \sqrt{\frac{k}{m}}$$

# LC Oscillations (2)

➔ Solution is same as mass on spring ⇒ oscillations

$$q = q_{max}\cos(\omega t + \theta) \qquad \omega = \sqrt{\frac{k}{m}}$$

- $q_{max}$ is the maximum charge on capacitor
- $\theta$ is an unknown phase (depends on initial conditions)

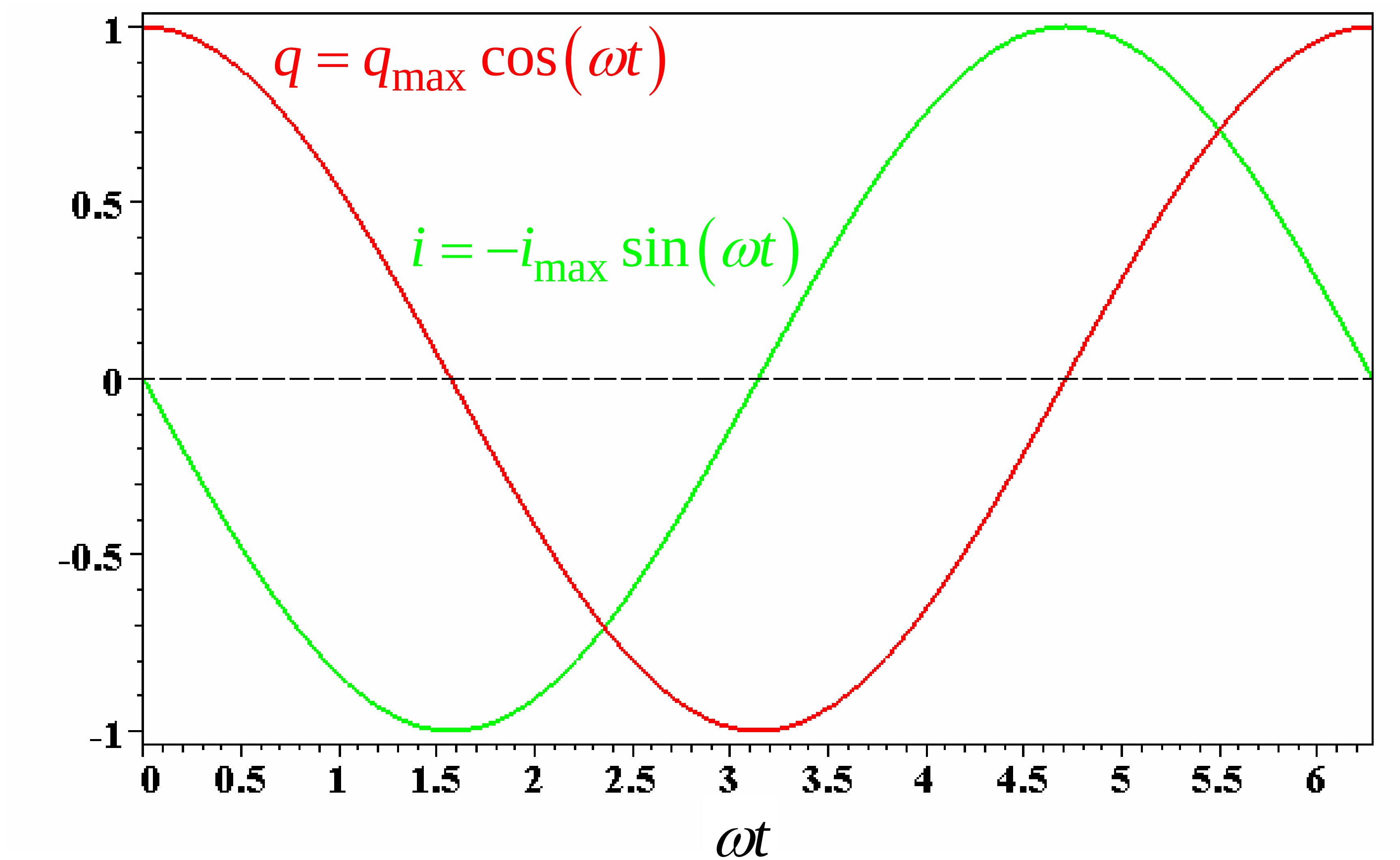
➔ Calculate current: i = dq/dt

$$i = -\omega q_{max}\sin(\omega t + \theta) = -i_{max}\sin(\omega t + \theta)$$

➔ Thus both charge and current oscillate

- Angular frequency $\omega$, frequency $f = \omega/2\pi$
- Period: $T = 2\pi/\omega$
- Current and charge differ in phase by 90°

# Plot Charge and Current vs t



# Energy Oscillations in LC Circuits

➔Total energy in circuit is conserved. Let's see why

$$L\frac{di}{dt}+\frac{q}{C}=0$$

Equation of LC circuit

$$L\frac{di}{dt}i+\frac{q}{C}\frac{dq}{dt}=0$$

Multiply by i = dq/dt

$$\frac{L}{2}\frac{d}{dt}\left(i^2\right)+\frac{1}{2C}\frac{d}{dt}\left(q^2\right)=0$$

Use $\frac{dx^2}{dt}=2x\frac{dx}{dt}$

$$\frac{d}{dt}\left(\tfrac{1}{2}Li^2+\tfrac{1}{2}\frac{q^2}{C}\right)=0$$

$$\tfrac{1}{2}Li^2+\tfrac{1}{2}\frac{q^2}{C}=\text{const}$$

$U_L + U_C = \text{const}$

# Oscillation of Energies

➔Energies can be written as (using $\omega^2 = 1/LC$)

$$U_C = \frac{q^2}{2C} = \frac{q_{\max}^2}{2C}\cos^2(\omega t+\theta)$$

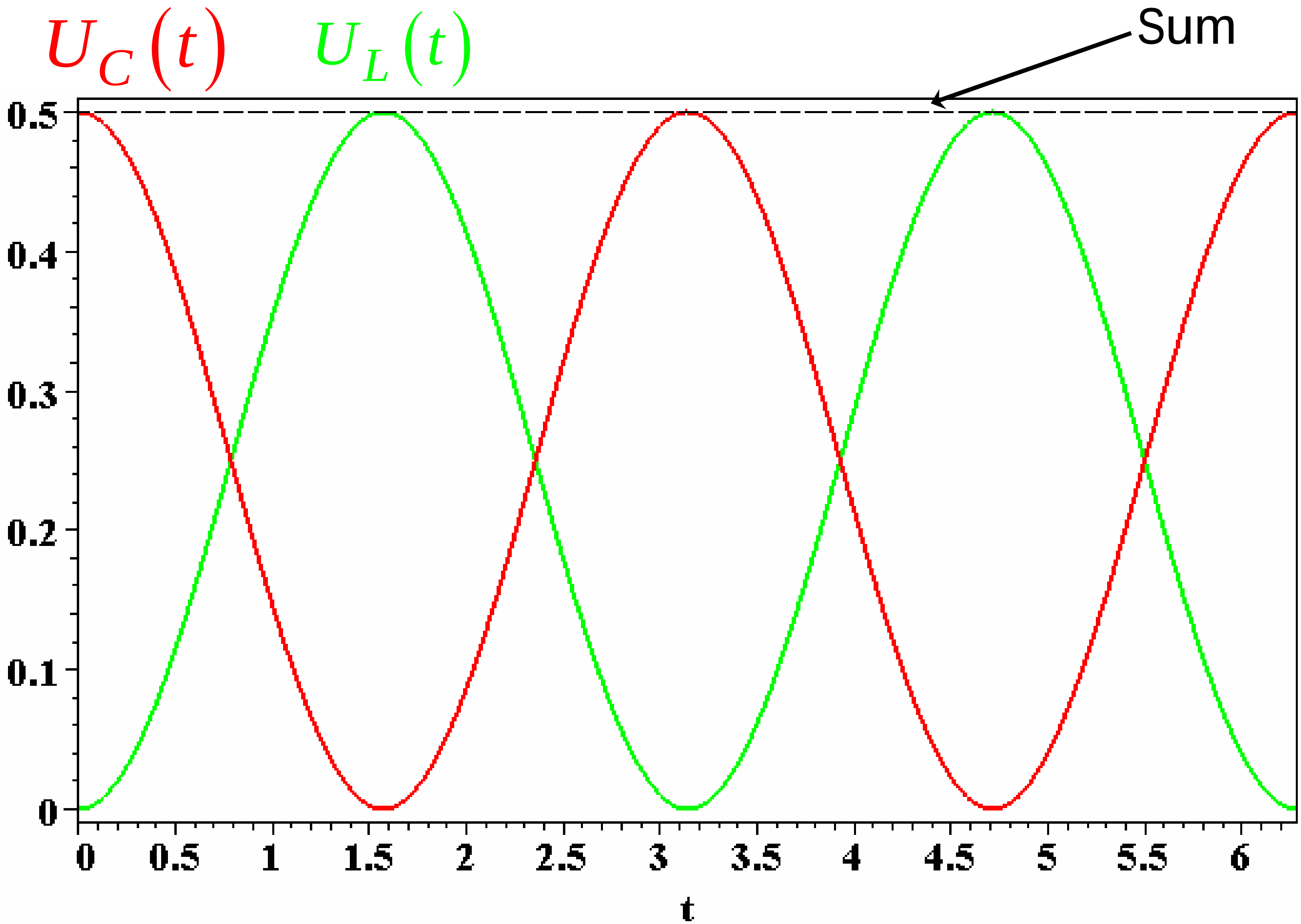
$$U_L = \tfrac{1}{2}Li^2 = \tfrac{1}{2}L\omega^2 q_{\max}^2 \sin^2(\omega t+\theta) = \frac{q_{\max}^2}{2C}\sin^2(\omega t+\theta)$$

➔Conservation of energy: $U_C + U_L = \dfrac{q_{\max}^2}{2C} = \text{const}$

➔Energy oscillates between capacitor and inductor

- Endless oscillation between electrical and magnetic energy
- Just like oscillation between potential energy and kinetic energy for mass on spring

# Plot Energies vs t



# LC Circuit Example

- Parameters
  - C = 20μF
  - L = 200 mH
  - Capacitor initially charged to 40V, no current initially
- Calculate ω, f and T
  - ω = 500 rad/s
  - f = ω/2π = 79.6 Hz
  - T = 1/f = 0.0126 sec

$$\omega = 1/\sqrt{LC} = 1/\sqrt{\left(2\times10^{-5}\right)\left(0.2\right)} = 500$$

- Calculate $q_{max}$ and $i_{max}$
  - $q_{max} = CV = 800\ \mu C = 8 \times 10^{-4}$ C
  - $i_{max} = \omega q_{max} = 500 \times 8 \times 10^{-4} = 0.4$ A
- Calculate maximum energies
  - $U_C = q^2_{max}/2C = 0.016$J   $U_L = Li^2_{max}/2 = 0.016$J

# LC Circuit Example (2)

➔Charge and current

$$q = 0.0008\cos(500t) \qquad i = \frac{dq}{dt} = -0.4\sin(500t)$$

➔Energies

$$U_C = 0.016\cos^2(500t) \qquad U_L = 0.016\sin^2(500t)$$

➔Voltages

$$V_C = q/C = 40\cos(500t)$$

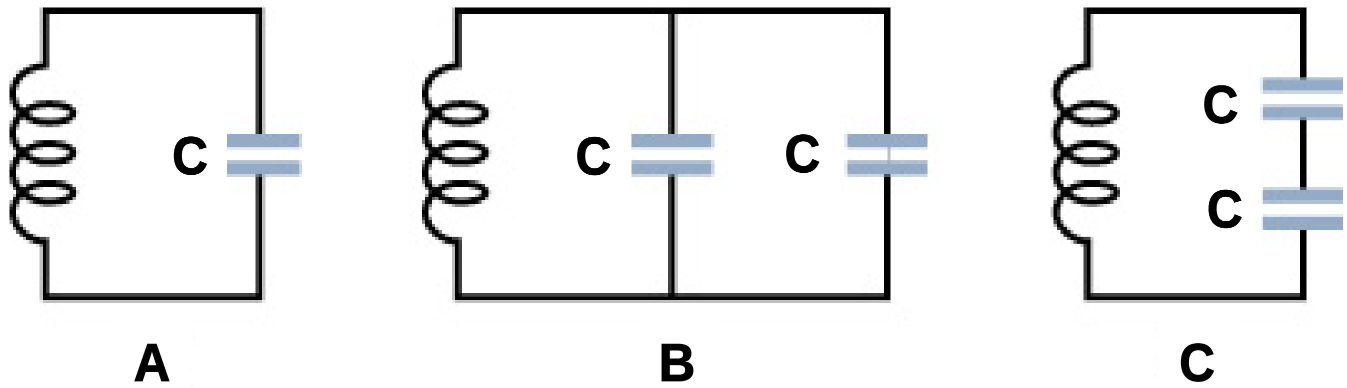
$$V_L = L\,di/dt = -L\omega i_{\max}\cos(500t) = -40\cos(500t)$$

➔Note how voltages sum to zero, as they must!

# Quiz

➔Below are shown 3 LC circuits. Which one takes the least time to fully discharge the capacitors during the oscillations?

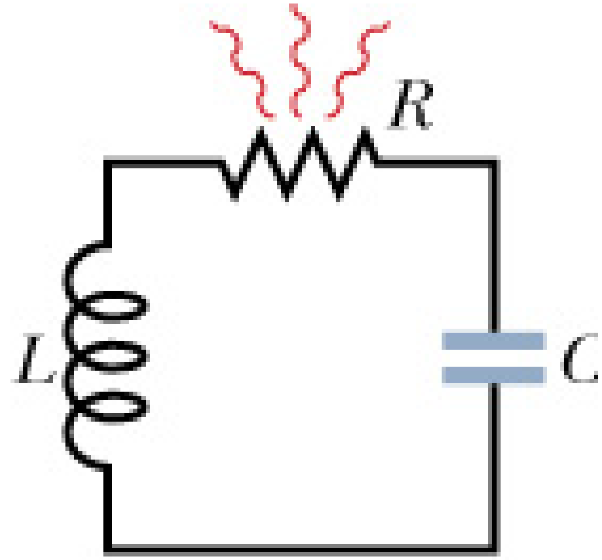
- ◆ (1) A
- ◆ (2) B
- ◆ (3) C




$$\omega = 1/\sqrt{LC}$$

C has smallest capacitance, therefore highest frequency, therefore shortest period

# RLC Circuit




➔The loop rule tells us

$$L\frac{di}{dt}+Ri+\frac{q}{C}=0$$

➔Use i = dq/dt, divide by L

$$\frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\frac{q}{LC}=0$$

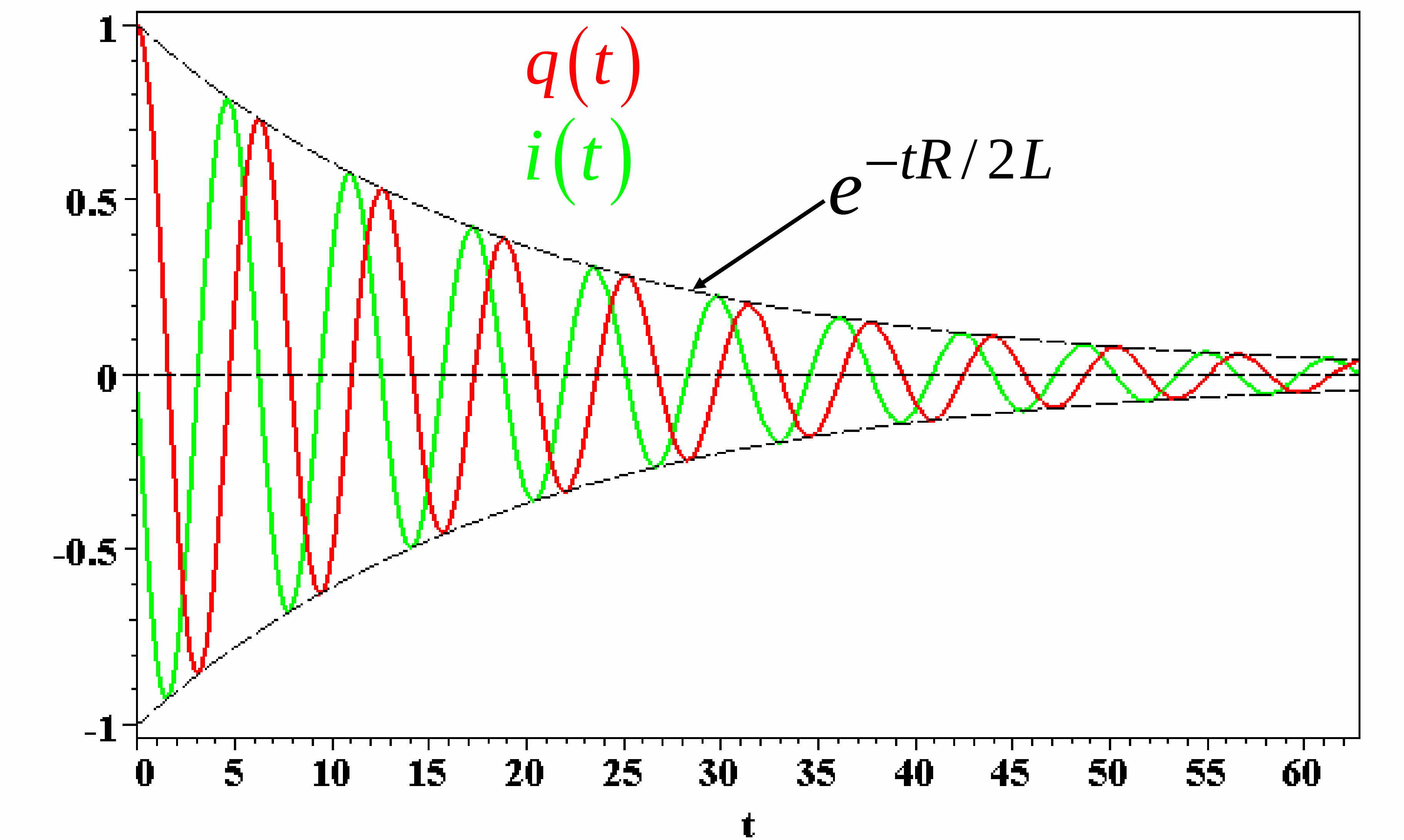
➔Solution slightly more complicated than LC case

$$q=q_{\max}e^{-tR/2L}\cos(\omega' t+\theta) \qquad \omega'=\sqrt{1/LC-(R/2L)^2}$$

➔This is a damped oscillator (similar to mechanical case)

- Amplitude of oscillations falls exponentially

# Charge and Current vs t in RLC Circuit



# RLC Circuit Example

- ➔ Circuit parameters
  - ◆ L = 12mL, C = 1.6μF, R = 1.5Ω
- ➔ Calculate ω, ω', f and T
  - ◆ ω = 7220 rad/s
  - ◆ ω' = 7220 rad/s
  - ◆ f = ω/2π = 1150 Hz
  - ◆ T = 1/f = 0.00087 sec

$$\omega = 1/\sqrt{(0.012)\left(1.6\times10^{-6}\right)} = 7220$$

$$\omega' = \sqrt{7220^2 - (1.5/0.024)^2} \simeq \omega$$

- ➔ Time for $q_{max}$ to fall to ½ its initial value $e^{-tR/2L} = 1/2$
  - ◆ t = (2L/R) * ln2 = 0.0111s = 11.1 ms
  - ◆ # periods = 0.0111/.00087 ≈ 13

# RLC Circuit (Energy)

$$L\frac{di}{dt}+Ri+\frac{q}{C}=0$$

Basic RLC equation

$$L\frac{di}{dt}i+Ri^2+\frac{q}{C}\frac{dq}{dt}=0$$

Multiply by i = dq/dt

$$\frac{d}{dt}\left(\tfrac{1}{2}Li^2+\tfrac{1}{2}\frac{q^2}{C}\right)=-i^2R$$

Collect terms
(similar to LC circuit)

$$\frac{d}{dt}\left(U_L+U_C\right)=-i^2R$$

$$U_{\text{tot}}\sim e^{-tR/L}$$

Total energy in circuit decreases at rate of $i^2R$ (dissipation of energy)

# Energy in RLC Circuit