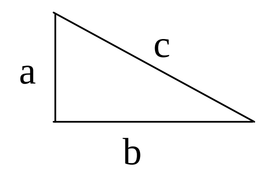
Math and Physics Refresher

This course assumes that you have studied Newtonian mechanics in a previous calculus-based physics course (i.e. PHY2048) and at least have co-registered in a vector calculus course (Calc 3). Listed below are some of the concepts in basic math, calculus, and physics that you are expected to know *or to acquire* during this course. This is not a complete summary of introductory math and physics. It is only meant to be a refresher of some of the concepts used in this course. Please report any inaccuracies to the professors.

Geometry



1. Pythagorean Theorem:
   The square of the hypotenuse of a right triangle is the sum of the squares of the two legs: $c^2 = a^2 + b^2$

2. Circumference of a circle: $C = 2\pi R$
3. Volume of a sphere: $V = \frac{4}{3}\pi R^3$
4. Surface area of sphere: $S = 4\pi R^2$

Calculus

1. Differentiation:

   You are expected to be able to take simple derivatives:

   $$\frac{d}{dx}x^n = nx^{n-1}$$

   $$\frac{d}{dx}\sin x = \cos x$$

   $$\frac{d}{dx}\cos x = -\sin x$$

   $$\frac{d}{dx}e^x = e^x$$

2. Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{dg}{dx} + g(x)\frac{df}{dx}$$

Example: $\frac{d}{dx}(x\sin x) = x\cos x + \sin x$

3. Chain Rule

$$\frac{d}{dx} f(g(x)) = \frac{df}{dg}\frac{dg}{dx}$$

Examples:

$$\frac{d}{dx}\sin 2x = 2\cos 2x$$

$$\frac{d}{dx}\exp(-x^2) = -2x\exp(-x^2)$$

4. Integration

You are expected to be able to perform simple integrals:

$$\int x^n dx = \frac{1}{n+1}x^{n+1}$$

$$\int dx\cos x = \sin x$$

$$\int dx\sin x = -\cos x$$

$$\int dx\, e^x = e^x$$

A purist would note that a constant should be added to these indefinite integrals.

5. Change of Variables

To use integration tables correctly, you must be able to change variables. For example:

$$I = \int_0^{\mathrm{L}} \sin^2\left(\frac{n\pi x}{L}\right)$$

Let $u = \frac{n\pi x}{L} \quad \Rightarrow \quad dx = \frac{L}{n\pi}du$

$$I = \frac{L}{n\pi}\int_0^{n\pi} du\sin^2 u$$

Then use $\int dx\sin^2 x = \frac{x}{2} - \frac{1}{4}\sin 2x$

$$I = \frac{L}{n\pi}\frac{n\pi}{2} = \frac{L}{2}$$

6. Integration By Parts

$$\int u\,dv = uv - \int v\,du$$

Approximations

For small *x*, the following expansions are useful:

1. $\dfrac{1}{1-x} \approx 1 + x + x^2 + \cdots$

2. Binomial Expansion: $(1+x)^n \approx 1 + nx + \cdots$

3. Taylor Expansion: $f(x) \approx f(0) + x \left.\dfrac{df}{dx}\right|_{x=0} + \dfrac{x^2}{2!} \left.\dfrac{d^2 f}{dx^2}\right|_{x=0} + \cdots$

Differential Equations

We will study the solutions to several differential equations when we study circuits in this class. Although you are not required to have taken a course in differential equations, we will learn how to solve the simplest ones:

1. The exponential function is the only function whose derivative is the function itself:
   $$\frac{df}{dx} = \alpha f(x)$$
   $f(x) = Ce^{\alpha x}$ is the general solution, where C and α are constants

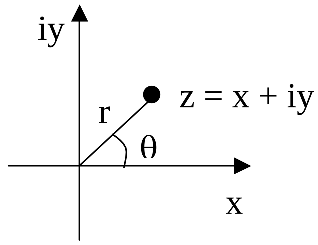
2. Two derivatives of the trigonometric functions give you back the same function with a sign change:
   $$\frac{d^2 f}{dx^2} = -k^2 f(x)$$
   $f(x) = A \sin kx + B \cos kx$ is the general solution, where $A, B,$ and k are constants

   Using complex exponentials (see next section), you can also represent this solution as:
   $$f(x) = A' e^{ikx} + B' e^{-ikx}$$

Complex Numbers

The analysis of circuits or electromagnetic waves in this class can be greatly simplified by the usage of complex numbers.

Complex (or imaginary) numbers are based on $i = \sqrt{-1}$. A complex number may be represented by $z = x + iy$, where x and y are real numbers. It can be represented by a point on a two-dimensional plane:



An alternative way to represent a complex number is $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

To find the magnitude of a complex number (the length r), multiply the number by its **complex conjugate**, then take the square root:

$$|z| = \sqrt{z^* z} = \sqrt{re^{-i\theta} re^{i\theta}} = r$$

or,

$$|z| = \sqrt{z^* z} = \sqrt{(x - iy)(x + iy)} = \sqrt{x^2 + y^2}$$

Note that to take the complex conjugate, replace i with $-i$

It is possible to represent the sine and cosine functions by complex exponentials:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Vectors (see also Appendix E in HRW 7/e)

- **Dot product**: projection of one vector along another

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = a_x b_x + a_y b_y + a_z b_z$$
$$|\mathbf{a}\cdot\mathbf{b}| = ab\cos\theta$$

- **Cross product**: product of vectors, with direction given by right-hand rule

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} = \det\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix} = \left(a_y b_z - b_y a_z\right)\mathbf{i} - \left(a_x b_z - b_x a_z\right)\mathbf{j} + \left(a_x b_y - b_x a_y\right)\mathbf{k}$$
$$|\mathbf{a}\times\mathbf{b}| = ab\sin\theta$$

Vector Calculus

The concepts of the following are useful, though not used heavily in PHY2049.

The gradient is the rate of change of a function along each direction:

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$
$$\mathbf{F} = grad\left(f\right) = \nabla f$$

It yields a vector function when applied to a scalar function.

The **Laplacian** operator is:

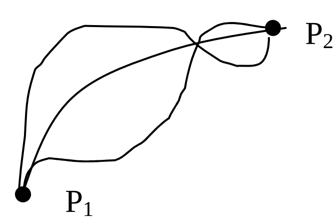
$$\nabla^2 \equiv \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

When applied to a scalar function, it yields a scalar result.

The **line integral** is the integral of a vector function projected along a one-dimensional path:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}\left[\mathbf{r}\left(t\right)\right]\cdot\frac{d\mathbf{r}}{dt}\,dt$$

The curve C is the path of function $\mathbf{r}(t)$, where t is a variable that parameterizes the path.



The **line integral** is the inverse of the gradient:

$$f(P_2) - f(P_1) = \int_{P1}^{P2} \nabla f\cdot d\mathbf{r}$$

This means that for conservative forces $\mathbf{F} = \nabla f$, it doesn't matter what path one takes to go from P1 to P2, the line integral is just the function evaluated at the endpoints. Also, if the path is a closed loop, the line integral equals zero.

The **surface integral** is the integral of a vector function projected onto a two-dimensional surface:

$$\int_S \mathbf{F} \cdot d\mathbf{s} = \int_S \mathbf{F}\left[\mathbf{r}(u,v)\right] \cdot \left(\frac{d\mathbf{r}}{du} \times \frac{d\mathbf{r}}{dv}\right) du\ dv$$

The surface *S* is parameterized by two variables, *u* and *v*.

The divergence of a vector function:

$$\nabla \cdot \mathbf{F} = div(\mathbf{F}) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

The divergence is the flux out of a volume *V*, per unit volume, as $V \to 0$:

$$\nabla \cdot \mathbf{F} \equiv \lim_{V \to 0} \frac{1}{V} \oint_S \mathbf{F} \cdot d\mathbf{A}$$

Divergence Theorem:

$$\int_V \nabla \cdot \mathbf{F}\ dV = \oint_S \mathbf{F} \cdot d\mathbf{A}$$

The curl of a vector function:

$$\nabla \times \mathbf{F} = curl(\mathbf{F}) = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}$$

The curl (when projected along the normal to a loop) is the **circulation** of a vector function around a closed loop of area *A*, per unit area, as $A \to 0$:

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \equiv \lim_{A \to 0} \frac{1}{A} \oint_C \mathbf{F} \cdot d\mathbf{s}$$

Stokes' Theorem:

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \oint_C \mathbf{F} \cdot d\mathbf{s}$$

Physics – Mechanics

Newton's Laws:

1. An object maintains constant velocity unless acted upon by an external force
2. The acceleration of an object is proportional to the applied external force divided by the mass of the object (the inertia)

   $$\mathbf{F} = m\mathbf{a}$$

   This is a vector equation. It can also be written as:

   $$\mathbf{F} = \frac{d\mathbf{p}}{dt} \qquad \text{where } \mathbf{p} = m\mathbf{v} \text{ is the momentum}$$

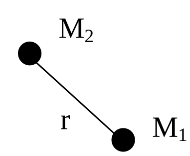
   This is your first differential equation! Note that $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2}$ also.
3. Force exerted by body 1 on body 2 is equal and opposite to the force that body 2 exerts on body 1. In other words, the force of gravity acting on you is balanced by an opposite force applied by the floor so that you do not fall to the center of the Earth!

If the force exerted on a body is constant, then so is the acceleration. In that case

$$\mathbf{v} = \mathbf{a}\, t$$

$$\mathbf{x} = \mathbf{v}t = \frac{1}{2}\mathbf{a}\, t^2$$

Some examples of forces:



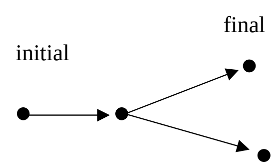
Gravitational force near the surface of the Earth: $F = mg$ where $g = 9.8 \text{ m} / \text{s}^2$

Newton's law of Gravity: $F = G\frac{m_1 m_2}{r^2}$ where $G = 6.6726 \times 10^{-11} \text{ N-m}^2 / \text{kg}^2$

Hooke's Law: $F = kx$ where $k =$ spring constant, $x =$ displacement

Conservation of Momentum:

Momentum is defined by $\mathbf{p} = m\mathbf{v}$. It is a vector quantity. The vector sum of the momentum of all particles before an interaction is the same as it is afterwards. In other words, it is conserved. For two masses which collide (but do not stick):



$$m_1 v_1 + m_2 v_2 = m_1 v_1{}' + m_2 v_2{}'$$

This is an elastic collision. If they do stick, it is inelastic. In that case:

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v$$

Work and Energy:

Applying a force over a distance requires work:

$$W = \mathbf{F} \cdot \mathbf{d}$$

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{s}$$

The work done is opposite to the change in the potential energy:

$$\Delta W = -\Delta U$$

The force is derived from the rate of change of the potential energy:

$$\mathbf{F} = -\nabla U$$

Conservation of Energy:

In Newtonian physics, we learn that the kinetic energy is conserved for elastic collisions (but not inelastic collisions, where some of the energy goes into mass!) The kinetic energy is given by

$$K = \frac{1}{2} m v^2$$

For the elastic collision of two objects, conservation of energy implies:

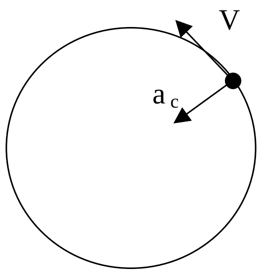
$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 {v_1'}^2 + \frac{1}{2} m_2 {v_2'}^2$$

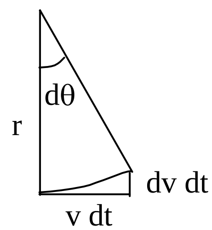
Circular motion:

In uniform circular motion, the magnitude of the velocity of an object is constant, though its components are not. The magnitude of the centripetal acceleration to achieve uniform circular motion is

$$a_c = \frac{v^2}{r}$$

The centripetal force responsible for this acceleration is just the mass times this quantity.





$$d\theta = \frac{v\ dt}{r} = \frac{dv\ dt}{v\ dt} = \frac{dv}{v}$$

$$\Rightarrow \frac{dv}{dt} \equiv a = \frac{v^2}{r}$$