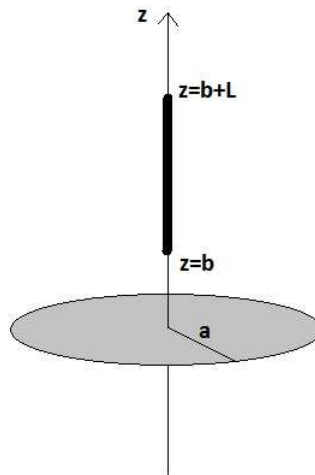


## Exam 1 Solution

As customary, choice (a) is the correct answer in all the following problems.

### Problem 1

A uniformly charged (thin) non-conducting rod is located on the central axis a distance  $b$  from the center of an uniformly charged non-conducting disk. The length of the rod is  $L$  and has a linear charge density  $\lambda$ . The disk has radius  $a$  and a surface charge density  $\sigma$ . The total force among these two objects is



- (1)  $\vec{F} = \frac{\lambda\sigma}{2\epsilon_0} \left( L + \sqrt{a^2 + b^2} - \sqrt{(b+L)^2 + a^2} \right) \hat{k}$
- (2)  $\vec{F} = \frac{\lambda\sigma a^2 L}{4\epsilon_0 b^2} \hat{k}$
- (3)  $\vec{F} = \frac{\lambda\sigma a^2 L}{8\epsilon_0 b^2} \hat{k}$
- (4)  $\vec{F} = \frac{\sigma}{2\epsilon_0} \left( 1 - \frac{L}{\sqrt{L^2 + a^2}} \right) \hat{k}$
- (5)  $\vec{F} = \frac{\lambda\sigma L^2}{2\epsilon_0 \sqrt{L^2 + a^2}} \hat{k}$

### Solution

We saw in class that the electric field created at any point along the central axis is given by

$$\vec{E}(z) = \frac{\sigma}{2\epsilon_0} \left( 1 - \frac{z}{\sqrt{a^2 + z^2}} \right) \hat{k}$$

Breaking up the rod into an infinite number of infinitesimally small point charges  $dq$ , we have that the net force on each tiny charge is  $d\vec{F} = dq\vec{E}(z)$ . Summing up all these

contributions, and using the fact that  $dq = \lambda dz$  gives

$$\vec{F} = \int dq \frac{\sigma}{2\epsilon_0} \left( 1 - \frac{z}{\sqrt{a^2 + z^2}} \right) \hat{k} \quad (1)$$

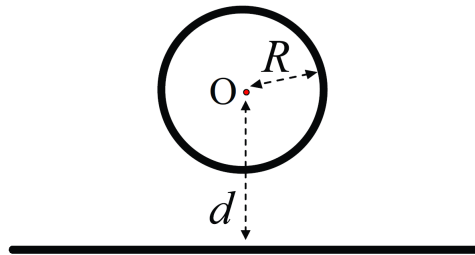
$$= \frac{\lambda\sigma}{2\epsilon_0} \hat{k} \int_b^{b+L} dz \left( 1 - \frac{z}{\sqrt{a^2 + z^2}} \right) \quad (2)$$

$$= \frac{\lambda\sigma}{2\epsilon_0} \hat{k} \left( z - \sqrt{a^2 + z^2} \right)_b^{b+L} \quad (3)$$

$$= \frac{\lambda\sigma}{2\epsilon_0} \left( L - \sqrt{a^2 + (b+L)^2} + \sqrt{a^2 + b^2} \right) \hat{k} \quad (4)$$

## Problem 2

A uniformly charged (thin) non-conducting shell (hollow sphere) of radius  $R$  with the total positive charge  $Q$  is placed at a distance  $d$  away from an infinite non-conducting sheet carrying a uniformly distributed positive charge with a density  $\sigma$ . The distance  $d$  is measured from shell's center (point O). What is the magnitude of the total electric field at the center of the shell?



- (1)  $\frac{\sigma}{2\epsilon_0}$  (2)  $\frac{Q}{4\pi\epsilon_0 R^2} + \frac{\sigma}{2\epsilon_0}$  (3)  $\frac{Q}{4\pi\epsilon_0 R^2} + \frac{\sigma}{\epsilon_0}$  (4)  $\frac{Q}{4\pi\epsilon_0 R^2}$  (5)  $\frac{Q}{4\pi\epsilon_0 R} + \frac{\sigma}{\epsilon_0}$

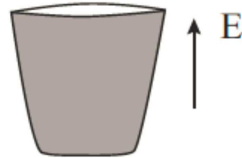
## Solution

This problem is very easy to solve if one recalls the superposition principle. The total electric field at any point in space is equal to the sum of the individual contributions from each source. The electric field produced by the sphere in its interior is always zero<sup>1</sup>. The electric field produced by a non-conducting infinitely long sheet is  $\frac{\sigma}{2\epsilon_0}$  everywhere in space. Therefore, the sum of these two contributions at the center of the sphere is simply  $\frac{\sigma}{2\epsilon_0}$ .

<sup>1</sup>This can be seen as a consequence of Gauss' Law for this spherically symmetric situation

### Problem 3

A round wastepaper basket with a 0.15 m radius opening is in a uniform electric field of 300 N/C, perpendicular to the opening. The total flux through the sides and bottom, in  $N \cdot m^2/C$ , is:



(1) - 21 (2) 4.2 (3) 0 (4) 280 (5) can't tell without knowing the areas of the sides and bottom

### Solution

Because the electric field is a constant everywhere, the electric flux through any closed surface is zero. Thus, we can write

$$\Phi = \oint \vec{E} \cdot d\vec{A} = 0$$

Decomposing the entire surface of the basket into the sides, bottom, and top, yields

$$\Phi = \oint \vec{E} \cdot d\vec{A} = \int_{\text{sides}} \vec{E} \cdot d\vec{A} + \int_{\text{bottom}} \vec{E} \cdot d\vec{A} + \int_{\text{top}} \vec{E} \cdot d\vec{A} = 0$$

thus

$$\int_{\text{sides}} \vec{E} \cdot d\vec{A} + \int_{\text{bottom}} \vec{E} \cdot d\vec{A} = - \int_{\text{top}} \vec{E} \cdot d\vec{A} \quad (5)$$

$$= -E A_{\text{top}} \quad (6)$$

$$= -E \pi r^2 \quad (7)$$

$$= -300 \pi 0.15^2 \quad (8)$$

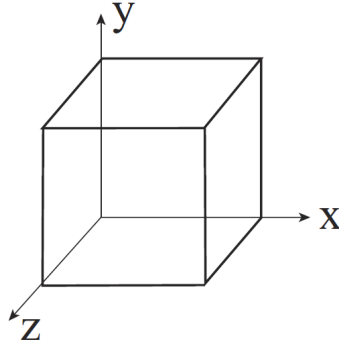
$$= -21 \quad (9)$$

Therefore

$$\int_{\text{sides}} \vec{E} \cdot d\vec{A} + \int_{\text{bottom}} \vec{E} \cdot d\vec{A} = -21 \text{ N/Cm}^2$$

**Problem 4**

An electric field given by  $\vec{E} = 10\hat{i} - 5(y^2 + 5)\hat{j}$  pierces the Gaussian cube of the figure, where the cube is 2 m on a side. (E is in newtons per coulomb and  $y$  is in meters.) What is the net electric flux through the entire cube?



- (1) -80 N/C m<sup>2</sup> (2) 80 N/C m<sup>2</sup> (3) 0 (4) 20 N/C m<sup>2</sup> (5) -20 N/c m<sup>2</sup>

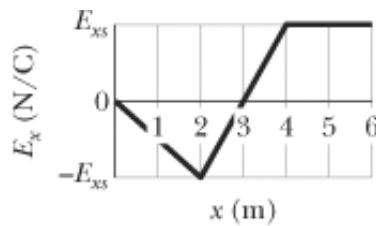
**Solution**

We can break up the entire closed surface into the six sides of the cube. The front and back sides do not contribute since  $\vec{E}$  lies on the  $x, y$  plane, thus it is parallel to these sides. Also, the  $x$  component  $E_x = 10$  of the electric field is constant. This implies that the contributions from this component would cancel in each other out among all sides. The same will happen with the constant part in  $E_y = -5y^2 - 25$ . Therefore, we end up with

$$\Phi = -5 \int_{y=2} y^2 dx dz + 5 \int_{y=0} y^2 dx dz = -5 \cdot (2)^2 \cdot A = -5 \cdot (2)^2 \cdot 2^2 = -80$$

**Problem 5**

A graph of the  $x$  component of the electric field as a function of  $x$  in a region of space is shown in the figure. The scale of the vertical axis is set by  $E_{xs} = 16.0$  N/C. The  $y$  and  $z$  components of the electric field are zero in this region. If the electric potential at the origin is 10 V, what is the electric potential (in V) at  $x = 4.0$  m?



- (1) 26 (2) -6 (3) 36 (4) 0 (5) 42

### Solution

By definition we have

$$V_b - V_a = - \int_a^b \vec{E} \cdot d\vec{s}$$

Where  $a$  and  $b$  are just two points in which we measure the electric potential  $V$ . Since we are given that  $V(x = 0) = 10$  V, we might as well use that as our point  $a$ , and  $x = 4$  as  $b$  to find  $V_b$ . Since the  $y$  and  $z$  components of the electric field are zero everywhere, we have  $\vec{E} \cdot d\vec{s} = E_x dx$ , and  $E_x$  is given by the plot as a function of  $x$ . Therefore, we can write

$$V_b = V_a - \int_a^b E_x dx \quad (10)$$

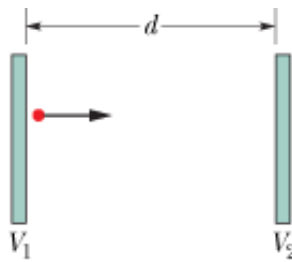
But the integral above is simply the area under the curve  $E_x$  vs.  $x$ . We just need to be careful with the sign of that area since it is negative in the range  $0 \leq x \leq 3$ , and positive in the range  $x > 4$ . Thus,

$$V(x = 4) = V(x = 0) - \int_0^4 E_x dx \quad (11)$$

$$= 10 - (-24 + 8) = 26 \quad (12)$$

### Problem 6

In the figure, a charged particle (either an electron or a proton; you need to find out which it is) is moving rightward between two parallel charged plates. The plate potentials are  $V_1 = -25$  V and  $V_2 = -35$  V. The particle is slowing down from an initial speed of  $3 \times 10^6$  m/s at the left plate. What is its speed, in m/s, just as it reaches plate 2?



- (1)  $2.4 \times 10^6$  (2)  $1.6 \times 10^6$  (3) not possible to know without knowing the plates separation  
(4)  $2.4 \times 10^{12}$  (5)  $3.5 \times 10^{12}$

### Solution

If we use conservation of energy (potential plus kinetic), this problem is really straightforward. The potential energy of the particle when it starts from plate 1 is  $U_1 = qV_1$  and

when it arrives at plate 2 is  $U_2 = qV_2$ . (Recall that the electric potential is a continuous function in space, therefore a particle very close to the plates will be at an electric potential equal to the one on the plate<sup>2</sup>). Therefore, conservation of energy reads

$$U_1 + K_1 = U_2 + K_2 \quad (13)$$

$$qV_1 + \frac{1}{2}mv_1^2 = qV_2 + \frac{1}{2}mv_2^2 \quad (14)$$

Solving for  $v_2$  gives

$$v_2 = \sqrt{v_1^2 + \frac{2q}{m}(V_1 - V_2)} \quad (15)$$

But now it comes a crucial point. What  $m$  do we use? The mass of a proton or that of an electron? This is very important since their masses differ by a factor of almost 2000! To clarify this, we recall that  $\vec{E} = -\vec{\nabla}V$ . This implies that the electric field, at a certain point in space, points in the **opposite** direction of  $\vec{\nabla}V$  at that location. Imagine we draw an  $x$  axis going from plate 1 to plate 2. Since the electric field is constant everywhere between the plates, i.e.  $\vec{E} = (E_x, 0, 0)$  where  $E_x = \text{constant}$ , the potential  $V(x)$  is a monotonic function of  $x$  in going from one plate to the other. Thus, since the potential at plate 1 is  $V_1 = -25$  V and decreases to  $V_2 = -35$  V at plate 2,  $\vec{\nabla}V$  is negative along the entire range of  $x$ . In other words,  $\vec{\nabla}V$  points to negative direction of  $x$  (left). Therefore, from  $\vec{E} = -\vec{\nabla}V$  we arrive at the conclusion that  $\vec{E}$  points in the positive direction on  $x$  (right). Now, since the particle is slowing down, the total force on it must be in the opposite direction of motion. Since the particle is traveling to the right, the net force ought to point to the left. From  $\vec{F} = q\vec{E}$ , we see that this is only possible if  $q$  is negative since  $\vec{E}$  points to the right. Therefore, the particle is an electron. Using then the electron's mass  $m_e = 9.1 \times 10^{-31}$  kgs, and charge  $q = -1.6 \times 10^{-19}$  C, we have

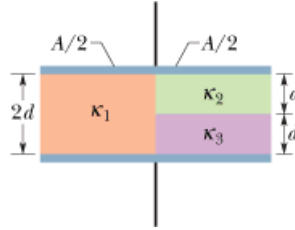
$$v_2 = \sqrt{v_1^2 + \frac{2q}{m}(V_1 - V_2)} = 2.4 \times 10^6 \text{ m/s} \quad (16)$$

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<sup>2</sup>Notice that since  $\vec{E} = -\vec{\nabla}V$ , the electric potential  $V$  must be a continuous function. Otherwise it would imply an infinite  $\vec{E}$ .

### Problem 7

The figure shows a parallel-plate capacitor of plate area  $A$  and plate separation  $2d$ . The left half of the gap is filled with material of dielectric constant  $\kappa_1 = 12$ ; the top of the right half is filled with material of dielectric constant  $\kappa_2 = 20$ ; the bottom of the right half is filled with material of dielectric constant  $\kappa_3 = 30$ . What is the capacitance in terms of  $\epsilon_0$ ,  $A$ , and  $d$ ?



- (1)  $9\frac{\epsilon_0 A}{d}$  (2)  $62\frac{\epsilon_0 A}{d}$  (3)  $18\frac{\epsilon_0 A}{d}$  (4)  $31\frac{\epsilon_0 A}{d}$  (5) none of these

### Solution

Assuming that the separation between the plates is much smaller than their extension, we can ignore fringe effects at the edges of the plates and at the junction of the two materials. Since the plates of a capacitor are made of conducting material, the plates are equipotential surfaces. This and the first statement allows us to consider this system as made of capacitor three capacitors  $C_1, C_2$  and  $C_3$  where  $C_2$  and  $C_3$  are in series, and this combination is in parallel with  $C_1$ . Thus

$$C_{\text{eq}} = C_1 + \frac{C_2 C_3}{C_2 + C_3} \quad (17)$$

With

$$C_1 = \kappa_1 \frac{\epsilon_0 A/2}{2d} \quad C_2 = \kappa_2 \frac{\epsilon_0 A/2}{d} \quad \text{and} \quad C_3 = \kappa_3 \frac{\epsilon_0 A/2}{d} \quad (18)$$

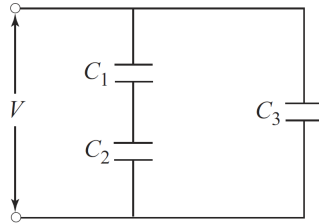
we have

$$C_{\text{eq}} = \frac{\epsilon_0 A}{2d} \left( \frac{\kappa_1}{2} + \frac{\kappa_2 \kappa_3}{\kappa_2 + \kappa_3} \right) \quad (19)$$

$$= 9 \frac{\epsilon_0 A}{d} \quad (20)$$

### Problem 8

In the figure shown, a potential difference of  $V = 10 \text{ V}$  is applied across the arrangement of capacitors with capacitances of  $C_1 = C_2 = 4\mu\text{F}$ , and  $C_3 = 6\mu\text{F}$ . What is the charge  $q_2$  on capacitor  $C_2$ ?



- (1)  $20\mu\text{C}$  (2)  $40\mu\text{C}$  (3)  $60\mu\text{C}$  (4)  $80\mu\text{C}$  (5)  $10\mu\text{C}$

### Solution

Capacitors  $C_1$  and  $C_2$  are in series, thus, share the same charge. If  $V_1$  and  $V_2$  are the voltages across each of them, we have

$$V = V_1 + V_2 \quad (21)$$

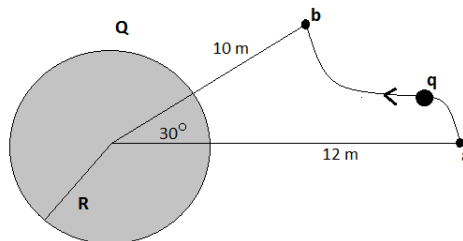
$$10 = Q/C_1 + Q/C_2 \quad (22)$$

Since  $C_1 = C_2 \equiv C = 4\mu\text{F}$ , we get

$$Q = \frac{CV}{2} = 20\mu\text{C} \quad (23)$$

### Problem 9

What is the minimum mechanical work that has to be done on the charge  $q = 1\mu\text{C}$  in order to bring it from point  $a$  to point  $b$ ? In figure, the solid sphere of charge  $Q = 2\mu\text{C}$  with a radius  $R = 2\text{m}$  is held fixed in space. Point  $a$  is located at  $12\text{m}$  from the center of the sphere and point  $b$  at  $10\text{m}$  as shown.



- (1)  $3 \times 10^{-4} \text{ J}$  (2)  $1.5 \times 10^{-4} \text{ J}$  (3)  $-3 \times 10^{-4} \text{ J}$  (4)  $-1.5 \times 10^{-4} \text{ J}$  (5)  $5.51 \times 10^{-5} \text{ J}$



### Solution

Since both charges have the same sign, and we are bringing the charge  $q$  closer to  $Q$ , we immediately know that we have to make a positive work due to the electric repulsion. Indeed, the work is

$$W = q(V_b - V_a) = q \left( \frac{kQ}{r_b} - \frac{kQ}{r_a} \right) \quad (24)$$

$$= 10^{-6} \times 9 \times 10^9 \times 2 \times 10^{-6} \left( \frac{1}{10} - \frac{1}{12} \right) \quad (25)$$

$$= 3 \times 10^{-4} \text{ J} \quad (26)$$

### Problem 10

The figure shows a non-conducting (thin) disk with a hole. The radius of the disk is  $b$  and the radius of the hole is  $a$ . A total charge  $Q$  is uniformly distributed on its surface. Assuming that the electric potential at infinity is zero, what is the electric potential at the center of the disk?



- (1)  $\frac{2kQ}{b+a}$  (2)  $\frac{2kQ}{b-a}$  (3)  $\frac{2kQ}{b^2-a^2}$  (4) 0 (5)  $\frac{kQ}{b^2}$

### Solution

We saw in class that the potential produced by a charged disk of radius  $R$ , at a distance  $z$  from it, along its central axis was

$$\frac{\sigma}{2\epsilon_0} \left( \sqrt{z^2 + R^2} - z \right) \quad (27)$$

By supersposition, we can think of the potential created by disk with the hole as the sum of two disks, with the same but opposite surface densities:

$$V(z) = \frac{\sigma}{2\epsilon_0} \left( \sqrt{z^2 + b^2} - z \right) - \frac{\sigma}{2\epsilon_0} \left( \sqrt{z^2 + a^2} - z \right) \quad (28)$$

$$= \frac{\sigma}{2\epsilon_0} \left( \sqrt{z^2 + b^2} - \sqrt{z^2 + a^2} \right) \quad (29)$$

Since we are only interested at the center, we have

$$V(0) = \frac{\sigma}{2\epsilon_0} (b - a) \quad (30)$$

The total area of the disk with the hole is  $A = \pi(b^2 - a^2)$ , thus

$$\sigma = \frac{Q}{\pi(b^2 - a^2)}$$

Finally

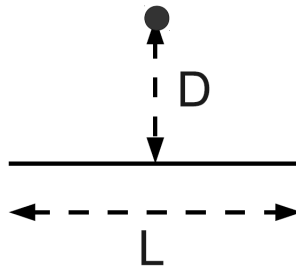
$$V(0) = \frac{1}{2\epsilon_0} (b-a) \frac{Q}{\pi(b^2 - a^2)} \quad (31)$$

$$= \frac{2kQ(b-a)}{(b-a)(b+a)} \quad (32)$$

$$= \frac{2kQ}{b+a} \quad (33)$$

### Problem 11

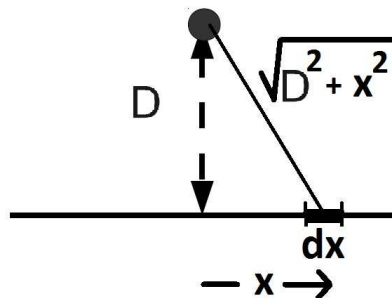
A wire segment of length  $L$  has constant linear charge density  $\lambda > 0$ . Which of the following expressions gives the magnitude of the electric field a distance  $D$  from the center of the wire (see figure)?



- (1)  $k\lambda D \int_{-L/2}^{L/2} \frac{dx}{(D^2 + x^2)^{3/2}}$       (2)  $k\lambda D \int_0^L \frac{dx}{\sqrt{D^2 + x^2}}$       (3)  $k\lambda D \int_0^L \frac{dx}{D^2 + x^2}$   
 (4) 0      (5)  $k\lambda D \int_{-L/2}^{L/2} \frac{dx}{D+x}$

### Solution

Putting the rod along the  $x$  axis with its center at the origin, the problem boils down to compute  $\vec{E}$  at the point  $(x, y) = (0, D)$



The contribution from the segment  $dx$  is

$$|d\vec{E}| = k \frac{dq}{D^2 + x^2} = k\lambda \frac{dx}{D^2 + x^2} \quad (34)$$

Also, we see immediately that the  $x$ -component of the total electric field will be zero due to mutual cancellations among mirror-symmetric segments of the rod. Thus, we only need the  $y$ -component, therefore

$$dE_y = |d\vec{E}| \frac{D}{\sqrt{D^2 + x^2}} = k\lambda D \frac{dx}{(D^2 + x^2)^{3/2}} \quad (35)$$

To get the total electric field, we simply add up all of these contributions, hence

$$E_y = \int dE_y = k\lambda D \int_{-L/2}^{L/2} \frac{dx}{(D^2 + x^2)^{3/2}} \quad (36)$$

### Problem 12

A charge  $Q$  is placed in the center of a shell of radius  $R$ . The flux of electric field through the shell surface is  $\Phi_0$ . What is the new flux through the shell surface, if its radius is doubled?

- (1)  $\Phi_0$  (2)  $2\Phi_0$  (3)  $4\Phi_0$  (4)  $\Phi_0/2$  (5)  $\Phi_0/4$

### Solution

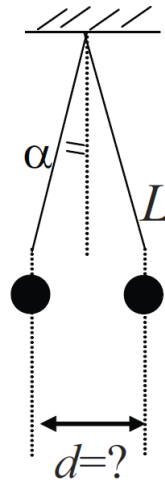
Gauss' law states that the electric flux through a closed surface is

$$\Phi \equiv \oint \vec{E} \cdot d\vec{s} = \frac{Q_{\text{enc}}}{\epsilon_0} \quad (37)$$

from where we see that the flux only cares about the total charge enclosed by the surface. By increasing the radius of the sphere we are merely increasing the size of the surface, but the enclosed charge remains the same. Therefore the new flux is just the old  $\Phi_0$ .

### Problem 13

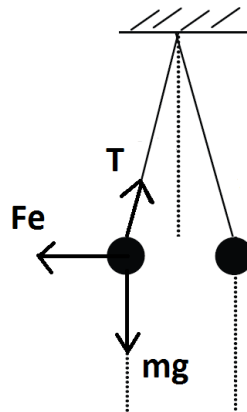
Two very small spheres have equal masses  $m$ , carry charges of the same sign and value  $q$ , and hang on strings of length  $L$  as shown in figure. Due to the repulsive force, the spheres are separated by some distance  $d$ . Find this distance. Assume that  $d \ll L$  so that you can use the approximation  $\tan \alpha \approx \sin \alpha \approx \alpha$



- (1)  $\sqrt[3]{2L \frac{q^2 k}{mg}}$  (2)  $\sqrt[3]{L \frac{q^2 k}{mg}}$  (3)  $\sqrt{2L \frac{q^2 k}{mg}}$  (4)  $\sqrt{L \frac{q^2 k}{mg}}$  (5)  $\sqrt{\frac{Lq^2 k}{2mg}}$

### Solution

To achieve equilibrium, we see from the figure that we need



$$T \cos \alpha = mg \quad \text{and} \quad T \sin \alpha = F_e \quad (38)$$

Now, combining these two equations as

$$\tan \alpha = \frac{F_e}{mg} \quad (39)$$

and using the Coulomb's force among the two small spheres  $F_e = kq^2/d^2$ , gives

$$\tan \alpha = \frac{kq^2}{d^2 mg} \quad (40)$$

Finally, for small values for  $\alpha$ ,  $\tan \alpha \approx \sin \alpha = \frac{d/2}{L}$ , we have

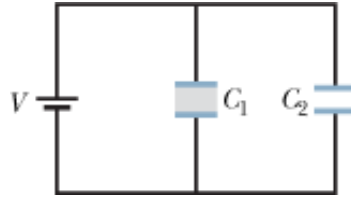
$$\frac{d/2}{L} \approx \frac{kq^2}{d^2 mg} \quad (41)$$

from where we obtain

$$d \approx \sqrt[3]{2L \frac{kq^2}{mg}} \quad (42)$$

### Problem 14

In figure, how much charge is stored on the parallel-plate capacitors by the 10 V battery? One is filled with air, and the other is filled with a dielectric for which  $\kappa = 2.0$ ; both capacitors have a plate area of  $2.00 \times 10^{-3} \text{ m}^2$  and a plate separation of 1.00 mm.



- (1) 0.53 nC (2) 0.35 nC (3) 1.06 nC (4) 0.53  $\mu\text{C}$  (5) 0.35  $\mu\text{C}$

### Solution

The charged stored on capacitor  $C_1$  is

$$q_1 = C_1 V = \kappa \frac{\epsilon_0 A}{d} V = 2 \frac{8.85 \times 10^{-12} \times 2.00 \times 10^{-3}}{10^{-3}} \times 10 = 3.5 \times 10^{-10}$$

and on capacitor  $C_2$  is

$$q_2 = C_2 V = \frac{\epsilon_0 A}{d} V = \frac{8.85 \times 10^{-12} \times 2.00 \times 10^{-3}}{10^{-3}} = 1.8 \times 10^{-10}$$

Thus, the total is

$$q_{\text{tot}} = 5.3 \times 10^{-10} = 0.53 \text{ nC}$$