Exam 2 Solution

PHY 2049 Summer '12

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Problem 1

The centripetal force is being provided by the magnetic force $\vec{F} = q\vec{v} \times \vec{B}$. In this case since $\vec{v} \perp \vec{B}$ $|\vec{F}| = qvB$, thus:

(1) $$qvB = \frac{mv^2}{r}$$ where r is the radius of the semi-circle.

Since a potential difference V is used to accelerate the particle, conservation of energy gives

$$qV = \frac{1}{2}mv^2$$ (the particle starts from rest)

(2) $$\Rightarrow mv^2 = 2qV$$

Putting this into (1):

$$\cancel{q}vB = \frac{2\cancel{q}V}{r} \rightarrow v = \frac{2V}{Br}$$

Since $x = 2r$, we have $v = \frac{4V}{Bx}$. Finally from (1), after cancelling one factor of v on each side, we have:

$$qB = \frac{m}{r} \cdot \frac{4V}{Bx}$$

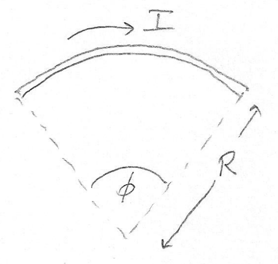
$$= \frac{m}{x/2} \cdot \frac{4V}{Bx}$$ which yields

$$\boxed{m = \frac{qB^2x^2}{8V}}$$

Problem 2

Recall that the magnetic field at the center of an arc carrying a current I is

$$|\vec{B}| = \frac{\mu_0 I \phi}{4\pi R}$$

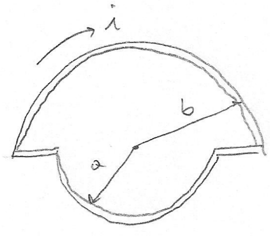


ϕ in radians.

Here we have two semi-circles, then

$$|\vec{B}|_{\text{semi-circle}} = \frac{\mu_0 I \pi}{4\pi R} = \frac{\mu_0 I}{4R}$$

For our problem:



The right hand rule tells you that the magnetic field due to, both, the upper and lower parts is in the direction INTO THE PAGE

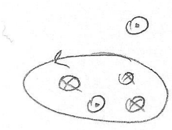
The straight segments do not contribute at the center due to $\vec{d\ell} \times \hat{r} = 0$ ($\vec{d\ell} \parallel \hat{r}$) (Biot–Savart). Therefore

$$\vec{B}_{TOTAL} = \left(\frac{\mu_0 i}{4b} + \frac{\mu_0 i}{4a} \right) \text{ into the page}$$

$$= \frac{\mu_0 i}{4} \left(\frac{1}{b} + \frac{1}{a} \right) \text{ into the page}$$

<u>Problem 3</u>

Ampere's Law reads



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

↳ enclosed current.

The arrow in the figure says that the integral is taken in the counterclockwise sense. This means that currents going <u>out of the page</u> are considered positive. Therefore

$$\oint \vec{B}\cdot d\vec{l} = \mu_0\left[\cancel{I}-\cancel{I}-I-I\right] = -2\mu_0 I$$

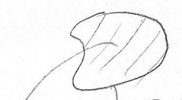
$$= -2\mu_0\cdot(2\text{ Amps}) = -4\mu_0 \quad //$$

Aside:

For the more skeptical ones:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \underbrace{\int \vec{J}\cdot d\vec{A}}_{\equiv I}$$

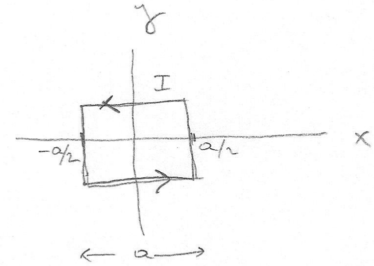
Using Stoke's theorem



$$\oint_{\Gamma} \vec{B}\cdot d\vec{l} = \int_{S(\Gamma)} (\vec{\nabla}\times\vec{B})\cdot d\vec{A} \longrightarrow$$

$d\vec{A}$ points in the direction given by right-hand rule (this is math, so no ambiguity here)

Problem 4

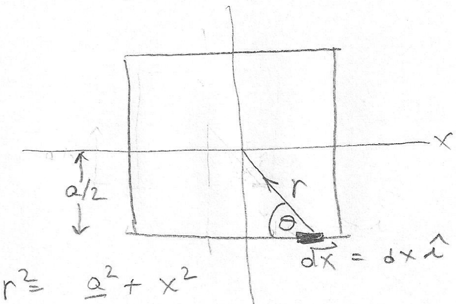


Just by looking at the figure, we know that all four sides are going to contribute with the same magnitude of $|\vec{B}|$ at the center.

For example, the bottom side, using Biot-Savart's Law gives:

$$\vec{B}_{bottom} = \frac{\mu_0 I}{4\pi} \int_{-a/2}^{a/2} \frac{\vec{dx} \times \hat{r}}{r^2}$$

$\vec{dx} = dx\,\hat{\imath}$



Pithagorean theorem $\longrightarrow$ $r^2 = \frac{a^2}{4} + x^2$

$\hat{r} = -\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$

where $\cos\theta = \frac{x}{r}$ & $\sin\theta = \frac{a/2}{r}$

$$\therefore\ \vec{dx} \times \hat{r} = dx\,\hat{\imath} \times \left[\cancel{-\frac{x}{r}\hat{\imath}} + \frac{a/2}{r}\hat{\jmath} \right] = \frac{a}{2r} dx\,\hat{k}$$

$\searrow \hat{\imath} \times \hat{\imath} = 0$ $\quad \searrow \hat{\imath} \times \hat{\jmath} = \hat{k}$

$$= \frac{a\,dx\,\hat{k}}{2\sqrt{a^2/4 + x^2}}$$

$$\therefore\ \vec{B}_{bottom} = \frac{\mu_0 I}{4\pi} \int_{-a/2}^{a/2} \frac{a\,dx\,\hat{k}}{2\sqrt{a^2/4 + x^2}\,(a^2/4 + x^2)} = \frac{\mu_0 I a \hat{k}}{8\pi} \int_{-a/2}^{+a/2} \frac{dx}{(a^2/4 + x^2)^{3/2}}$$

Since $\vec{B}_{TOTAL} = 4\vec{B}_{bottom}$

$$\Rightarrow \boxed{\vec{B}_{TOTAL} = \frac{\mu_0 I a \hat{k}}{2\pi} \int_{-a/2}^{+a/2} \frac{dx}{(a^2/4 + x^2)^{3/2}}}$$

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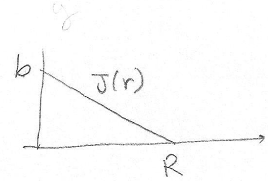
Problem 5

From the plot we read-off



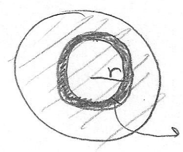
$$J(r) = b - \frac{b}{R}r = b\left(1 - \frac{r}{R}\right)$$

(You can see this recalling the straight line equation $y = b + mx$, where m is the slope which in this case is $m = -\frac{b}{R}$)

We want to use:

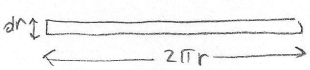
$$I = \int \vec{J} \cdot \vec{dA}$$

Inside the wire



width $= dr$

area $= 2\pi r\,dr$



$|\vec{dA}| = 2\pi r\,dr$ choose $\vec{dA}$ to go parallel to the current

$$I = \int_0^R b\left(1 - \frac{r}{R}\right) 2\pi r\,dr$$

$$= 2\pi b \int_0^R \left(1 - \frac{r}{R}\right) r\,dr = 2\pi b \left[\int_0^R r\,dr - \frac{1}{R}\int_0^R r^2\,dr\right]$$

$$= 2\pi b \left[\frac{R^2}{2} - \frac{1}{\cancel{R}} \cdot \frac{R^{\cancel{3}}}{3}\right] = 2\pi b \left[\frac{R^2}{2} - \frac{R^2}{3}\right] = 2\pi b R^2 \left[\frac{3-2}{6}\right]$$

$$\boxed{I = \frac{\pi b R^2}{3}}$$

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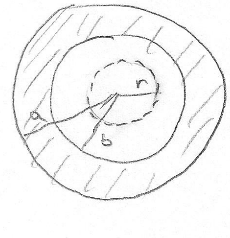
Problem 6

The problem has cylindrical symmetry. We can take advantage of this and use ampère's law:



$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$$

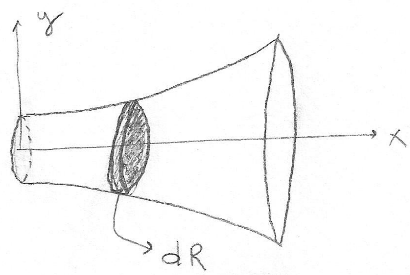
$$|\vec{B}|\, 2\pi r = \mu_0 I_{enc}$$

But inside the hollow part there's no current, ie. $I_{enc} = 0$, thus

$$\boxed{\vec{B} = 0}$$ //

Problem 7

Since resistors in series simply add up, we can think of this as a wire made up of many infinitesimally thin pieces of wire



Thus, the total resistance R will be $R = \int dR$.

We know that, for a wire:

$R = \dfrac{\rho d}{A}$ A: Area
d: length of the wire
ρ: resistivity of the wire

In our case:

$$dR = \frac{\rho\, dx}{A}$$

The area of this disk is $A = \pi r^2 = \pi y^2 = \pi\,(1+2x^2)^2$

Therefore:

$$R = \int dR = \int_0^2 \frac{\rho\, dx}{\pi\,(1+2x^2)^2}$$

$$R = \frac{\rho}{\pi} \int_0^2 \frac{dx}{(1+2x^2)^2}$$

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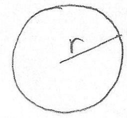
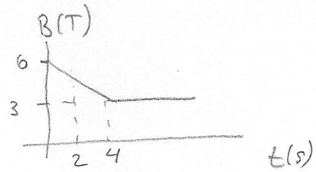
Problem 8

$$|EMF| = \frac{d\Phi_B}{dt}$$





$$= \frac{d}{dt}(BA) = \pi r^2 \frac{dB}{dt}$$

At $t = 2$ secs $\quad \frac{dB}{dt} = \frac{3-6}{4-0} = -\frac{3}{4}$

$$\therefore \quad |EMF| = \pi \cdot \cancel{2}^{2} \cdot \frac{3}{\cancel{4}} = 3\pi = 9.4 \text{ Volts}$$

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Problem 9

Ohm's law is $E = \rho J$

↳ electric field inside the material

Thus, in the three sections of different radii

$$\rho = \frac{E_1}{J_1} = \frac{E_2}{J_2} = \frac{E_3}{J_3}$$

Current conservation on the other hand says:

$$I = JA = J_1 A_1 = J_2 A_2 = J_3 A_3 \implies \frac{J_1}{J_2} = \frac{A_2}{A_1}$$

and so on.

Combining these two equations we have:

$$\frac{E_1}{E_2} = \frac{A_2}{A_1} \; ; \; \frac{E_2}{E_3} = \frac{A_3}{A_2} \implies \frac{E_1}{E_2} = \frac{\pi r_2^2}{\pi r_1^2} \; ; \; \frac{E_2}{E_3} = \frac{\pi r_3^2}{\pi r_2^2}$$

Take the second one. From the graph we read off E_1, E_2, E_3:

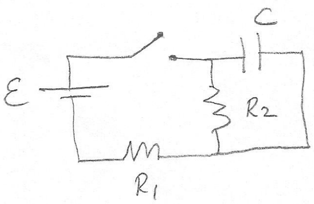
$$r_2 = r_3 \sqrt{\frac{E_3}{E_2}} = 3\,\text{mm} \sqrt{\frac{3}{8}} = 1.84\,\text{mm} \; //$$

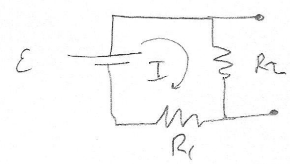
and from $\frac{E_1}{E_2} = \frac{r_2^2}{r_1^2} \rightarrow r_1 = r_2 \sqrt{\frac{E_2}{E_1}} = 1.84\,\text{mm} \sqrt{\frac{8}{5}}$

Therefore $\boxed{r_1 = 2.33\,\text{mm}}$ //

Problem 10

When the switch is closed and left there for a long time, the circuit looks like this:





because when the capacitor gets fully charged, no more current circulates to it. The current I is

$$I = \frac{\mathcal{E}}{R_1+R_2} = \frac{112}{6\times10^3 + 12\times10^3} = 6.7\times10^{-4}\text{ Amps}$$

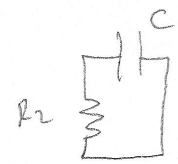
The voltage across R_2 (which is the same as the one across the capacitor) is:

$$V_{R_2} = IR_2 = 6.7\times10^{-4}\cdot 12\times10^3 = 8.04\text{ Volts}$$

The charge in the capacitor is then

$$Q = CV_{R_2} = 0.5\mu F\cdot 8.04\,V = 4.02\,\mu C$$

Now we the switch is opened, the circuit looks like:



which is an RC series circuit with a discharging capacitor. The charge on it is given by $Q(t) = Q_0 e^{-t/\tau}$

where Q_o is the initial charge $Q_o = 4.02\,\mu C$.

$$\tau = R_2 C = 12\times10^3 \cdot 0.5\times10^{-6} = 6\times10^{-3}$$

Therefore, at $t = 3.6$ ms

$$Q(3.6\,ms) = 4.02\times10^{-6}\, e^{-\left(\frac{3.6\times10^{-3}}{6\times10^{-3}}\right)}$$

$$= 2.2\times10^{-6}$$

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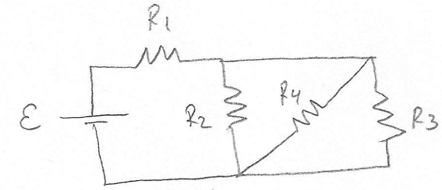
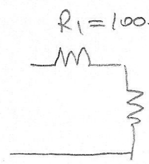
Problem 11



The equivalent resistance is:

- R_2, R_4, R_3 are in parallel



$$\left(\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}\right)^{-1} = \left(\frac{1}{30}+\frac{1}{30}+\frac{1}{30}\right)^{-1} = \left(\frac{3}{30}\right)^{-1} = 10$$

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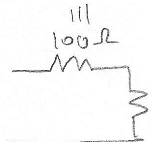


$\equiv$

$R_{eq} = 110\,\Omega$

The total current drained from the battery is then:

$$I = \frac{\mathcal{E}}{R_{eq}} = \frac{11}{110} = 0.1 \text{ Amps}$$

All of this current goes through R_1. Since $R_2 = R_3 = R_4$, this current splits evenly among the three resistors R_2, R_3 and R_4, thus this current is

$$\frac{0.1}{3} = 0.03 \text{ Amps}$$ //

Problem 12

The force on a current-carrying wire of length L, is given by

$$\vec{F} = i\,\vec{L} \times \vec{B}$$

where $\vec{L}$ points in the direction of the current. Here

$$\vec{L} = (\Delta x, \Delta y, \Delta z) = (3-1, 1-2, 0-0)$$

$$= (2,-1,0) = 2\hat{\imath} - \hat{\jmath}$$

Since $\vec{B} = 10\hat{k}$, we have

$$\vec{F} = i\vec{L} \times \vec{B} = 2 \cdot (2\hat{\imath} - \hat{\jmath}) \times 10\hat{k} = 20\,(2\hat{\imath} \times \hat{k} - \hat{\jmath} \times \hat{k})$$

$$= 20(-2\hat{\jmath} - \hat{\imath}) = -20\hat{\imath} - 40\hat{\jmath}$$

$$\therefore \quad \vec{F} = -20\hat{\imath} - 40\hat{\jmath} \;//$$

Problem 13

The potential energy is given by $U = -\vec{\mu} \cdot \vec{B}$.
The magnetic dipole moment is $\vec{\mu} = IA\hat{n}$, where
$\hat{n} = 0.06\hat{\imath} + 0.08\hat{\jmath}$, $I = 0.1$ and $A = \pi r^2 = \pi (0.04)^2$.
Since $\vec{B} = 0.5\hat{\imath} - 0.6\hat{\jmath}$, we have

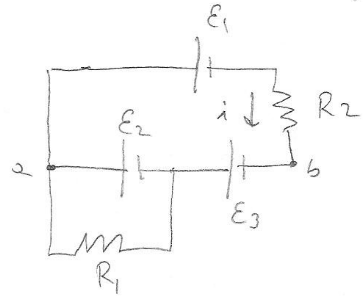
$$U = -\vec{\mu} \cdot \vec{B} = -IA\hat{n} \cdot \vec{B} = -IA(0.06\hat{\imath} + 0.08\hat{\jmath}) \cdot (0.5\hat{\imath} - 0.6\hat{\jmath})$$

$$= -IA(0.06 \times 0.5 - 0.08 \times 0.6)$$

$$= -0.1 \cdot \pi \cdot (0.04)^2 \cdot (0.03 - 0.048)$$

$$= -9.05 \times 10^{-6} \text{ Joules}$$ which is not in any of the choices, therefore the correct answer was : "None of these".

Problem 14



The loop rule applied to the upper loop reads:

$$-\mathcal{E}_1 - iR_2 + \mathcal{E}_3 + \mathcal{E}_2 = 0$$

From where we obtain:

$$i = \frac{\mathcal{E}_3 + \mathcal{E}_2 - \mathcal{E}_1}{R_2} = \frac{3+6-12}{30} = \frac{-3}{30} = -0.1$$

$\therefore \quad i = 0.1$ Amps //

Notice that the lower resistor plays no role in the value of the current through R2.