PHY2053 Lecture 16

Ch. 8.1-8.3: Rotational Kinetic Energy, Rotational Inertia, Torque
Key Concepts

- previously, we have found that a single point (CM) can be used to describe the bulk kinematic properties of an object
- now we will discuss one of the common modes of motion within the system (object) - rotation
- for the special case of rotation around a fixed axis, the kinetic energy due to rotation can be rephrased
- introduce the concept of moment of inertia
- this will introduce a set of equations similar to Newton’s laws, but for angular / rotational variables
Rotational Kinetic Energy

- Special case of a system of objects rotating around a common axis. For every object $i$, $v_i = \omega r_i$.

$$K_{tot} = \sum_{i=1}^{N} \frac{m_i v_i^2}{2} = \sum_{i=1}^{N} \frac{m_i (r_i \omega)^2}{2} = \frac{\omega^2}{2} \sum_{i=1}^{N} m_i r_i^2$$

- By analogy with $K_{mech} = m \frac{v^2}{2}$, where $m$ tells us about the inertia of the system, and $v$ about its velocity, define **moment of inertia** (mass equiv. for rotations):

$$I = \sum_{i=1}^{N} m_i r_i^2$$

$$K_{rot} = I \frac{\omega^2}{2}$$
### Tabulated Moments of Inertia for axes through CM

#### Table 8.1: Rotational Inertia for Uniform Objects with Various Geometrical Shapes

<table>
<thead>
<tr>
<th>Shape</th>
<th>Axis of Rotation</th>
<th>Rotational Inertia</th>
<th>Shape</th>
<th>Axis of Rotation</th>
<th>Rotational Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thin hollow cylindrical shell (or hoop)</td>
<td>Central axis of cylinder</td>
<td>$MR^2$</td>
<td>Solid sphere</td>
<td>Through center</td>
<td>$\frac{2}{3}MR^2$</td>
</tr>
<tr>
<td>Solid cylinder (or disk)</td>
<td>Central axis of cylinder</td>
<td>$\frac{1}{2}MR^2$</td>
<td>Thin hollow spherical shell</td>
<td>Through center</td>
<td>$\frac{2}{3}MR^2$</td>
</tr>
<tr>
<td>Hollow cylindrical shell or disk</td>
<td>Central axis of cylinder</td>
<td>$\frac{1}{2}Ma^2 + \frac{1}{2}mb^2$</td>
<td>Thin rod (or rectangular plate)</td>
<td>Perpendicular to rod through end (or along edge of plate)</td>
<td>$\frac{1}{3}ML^2$</td>
</tr>
<tr>
<td>Rectangular plate</td>
<td>Perpendicular to plate through center</td>
<td>$\frac{1}{12}Ma^2 + \frac{1}{12}mb^2$</td>
<td>Thin rod (or rectangular plate)</td>
<td>Perpendicular to rod through center (or parallel to edge of plate through center)</td>
<td>$\frac{1}{12}ML^2$</td>
</tr>
</tbody>
</table>
Parallel and Perpendicular Axis Theorems

• one can use the tabulated values to compute momenta of inertia of more complicated objects
• the tools used in that case are
  • parallel axis theorem - relates the momentum of inertia wrt an arbitrary axis of rotation to the momentum of inertia through the CM (tabulated)
  • perpendicular axis theorem - for a thin object relates the momentum of inertia around an axis perpendicular to the plane of the object to the two axes which are in the plane of the object
Discussion: Parallel Axis Theorem

The parallel axis theorem makes it possible to compute the moment of inertia of a complex object around a random axis, if one can compute the moment of inertia for the same object around a parallel axis which goes through the CM. Recall the definition of the moment of inertia, for a random axis.
(The distances $r_i$ represent the distances from the two axes - the axis of interest and the parallel axis which goes through the Center of Mass.

\[
I_{\text{axis}} = \sum_i m_i \vec{r}_i^2 = \sum_i m_i (\vec{r}_{\text{CM}} + \vec{r}_i')^2
\]

\[
I_{\text{axis}} = \sum_i m_i (\vec{r}_{\text{CM}})^2 + 2 \sum_i m_i \vec{r}_{\text{CM}} \cdot \vec{r}_i' + \sum_i m_i \vec{r}_i'^2
\]

\[
total \text{ mass of the system}
\]

\[
\text{Moment of inertia wrt CM}
\]
**Discussion: Parallel Axis Theorem 2**

Define: \( m_{\text{TOT}} = \sum_i m_i \)

\[ I_{\text{CM}} = \sum_i m_i (\hat{r}^i_i)^2 \]

Finally, compute: \( \hat{r}_{\text{CM}} \cdot \left( \sum_i m_i \hat{r}^i_i \right) \)

\[ \frac{m_{\text{TOT}}}{m_{\text{TOT}}} \cdot \hat{r}_{\text{CM}} \cdot \left( \sum_i m_i \hat{r}^i_i \right) = m_{\text{TOT}} \hat{r}_{\text{CM}} \cdot \left( \sum_i m_i \hat{r}^i_i \right) \]

\[ \frac{\sum m_i \hat{r}^i_i}{\sum m_i} \]

This is the position of the CM computed with respect to the CM (recall, the \( r^i_i \) are distances with respect to the Center of Mass). The position of the CM with respect to the CM is a zero vector, so the middle term is zero, we are left only with:

\[ I_{\text{axis}} = (\sum_i m_i) (\hat{r}_{\text{CM}})^2 + 2 \hat{r}_{\text{CM}} \cdot \sum_i m_i \hat{r}^i_i + \sum_i m_i (\hat{r}^i_i)^2 \]

\[ I_{\text{axis}} = m_{\text{TOT}} (\hat{r}_{\text{CM}})^2 + 2 \hat{r}_{\text{CM}} \cdot \sum_i m_i \hat{r}^i_i + I_{\text{CM}} \]

“The moment of inertia wrt an arbitrary axis is the sum of \( I_{\text{CM}} \) (the moment of inertia wrt a parallel axis through the CM) and \( m_{\text{TOT}} r_{\text{CM}}^2 \), where \( m_{\text{TOT}} \) is the mass of the system and \( r_{\text{CM}} \) is the distance of the CM wrt the axis.”
Discussion: Perpendicular Axis Theorem

Moment of inertia w.r.t. y-axis:
\[ I_y = \sum_i m_i x_i^2 \]

Moment of inertia w.r.t. x-axis:
\[ I_x = \sum_i m_i y_i^2 \]

Moment of inertia w.r.t. z-axis:
\[ I_z = \sum_i m_i (x_i^2 + y_i^2) \]
\[ = \sum_i m_i x_i^2 + \sum_i m_i y_i^2 \]
\[ = I_y + I_x \]
Example: Dumbbell Weight Moment Of Inertia

The dumbbell above consists of two homogenous, solid spheres, each of mass $M$ and radius $R$. The spheres are connected by a thin, homogenous rod of mass $m$ and length $L$. The entire dumbbell is rotating around the center of the rod. What is the moment of inertia of the dumbbell with respect to that axis?
Discussion: Dumbbell Weight Moment Of Inertia

The system can be broken into three pieces: two spheres, each of radius R, and a rod, rotating around its center.

\[ I_{CM}^{SPHERE} = \frac{2}{5}MR^2 \]

but the spheres are rotating around the center of the rod, not around their CMs. So, for each sphere,

\[ I_{AXIS}^{SPHERE} = I_{CM}^{SPHERE} + MR_{CM}^2 \]

\[ r_{CM} = \frac{L}{2} + R \]

\[ I_{AXIS}^{SPHERE} = \frac{2}{5}MR^2 + M\left(\frac{L}{2} + R\right)^2 \]

The rod is rotating around its center, so

\[ I_{CM}^{ROD} = \frac{ML^2}{12} \]

The total moment of inertia is then the sum of
Discussion: Dumbbell Weight Moment Of Inertia (2)

all the elements:

\[ I_{\text{TOTAL}} = 2 \cdot I_{\text{SPHERE}} + I_{\text{ROD}} \]

\[ = 2 \cdot \left[ \frac{2}{5} MR^2 + M \left( \frac{L}{2} + R \right)^2 \right] + \frac{ML^2}{12} \]

\[ = \frac{4}{5} MR^2 + 2 \frac{ML^2}{4} + 2MLR + 2MR^2 + \frac{ML^2}{12} \]

\[ = \frac{14}{5} MR^2 + 2MLR + \frac{7}{12} ML^2 \]
Unusual Atwood’s Machine

In the machine below, the small weight (mass m) is connected via an ideal pulley to the massless horizontal wheel of radius r. Ignore the moments of inertia of the connecting rods; the two masses M are both R away from the axis of rotation. What is the velocity of mass m as it falls h from its initial position, where it was at rest?
Discussion: Unusual Atwood’s Machine

Utilize $E_i = E_f$  \quad $E_i = K_i + U_i \rightarrow U_i = mgh$

$U_f = 0$

$E_f = U_f + K_f = 0 + \frac{m\nu^2}{2} + \frac{I\omega^2}{2}$, however $\nu = \omega r \quad \omega = \frac{\nu}{r}$

$\omega = \frac{\nu}{r}$

radius of pulley

$E_f = \frac{m\nu^2}{2} + \frac{2MR^2}{2}\left(\frac{\nu}{r}\right)^2$

$= \frac{m\nu^2}{2} + MR^2 \frac{\nu^2}{r^2} = \frac{m\nu^2}{2} + M\frac{R^2}{r^2} \nu^2 = \nu^2\left[\frac{m}{2} + M\frac{R^2}{r^2}\right]$

$E_i = E_f \rightarrow U_i = K_f \rightarrow mgh = \nu^2\left[\frac{m}{2} + M\frac{R^2}{r^2}\right]$ \quad $\text{mgh} = m\nu^2\left[\frac{1}{2} + \frac{MR^2}{mr^2}\right]$

$\nu^2 = \frac{gh}{\frac{1}{2} + \frac{MR^2}{mr^2}} \quad \rightarrow \quad \nu = \sqrt{\frac{gh}{\frac{1}{2} + \frac{MR^2}{mr^2}}}$
Next Lecture

Ch. 8.4, 8.6 (skip 8.5) Torque, Rotational Equilibrium, Rotational Formulation of Newton’s Laws