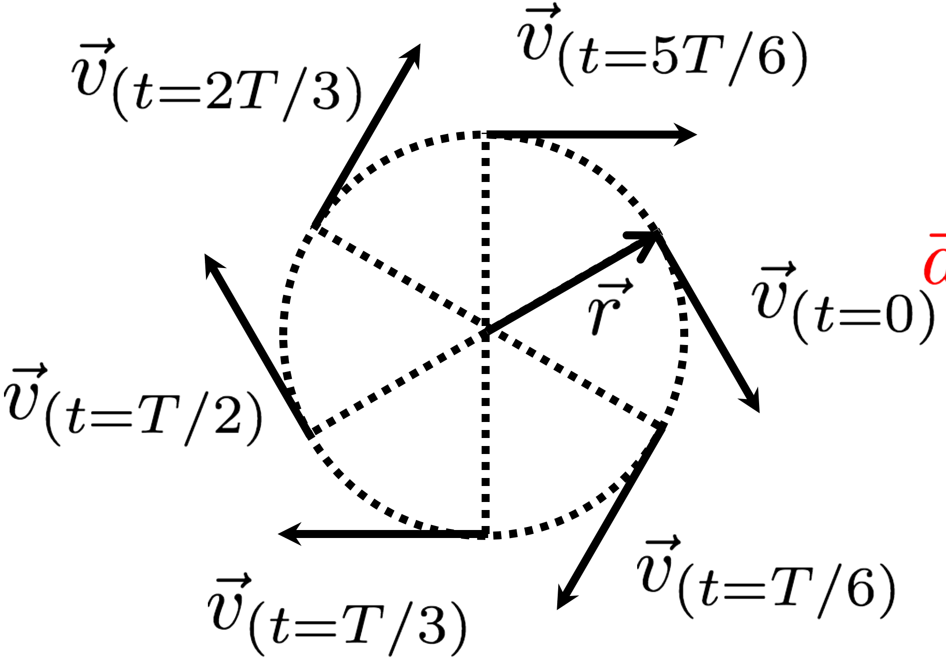
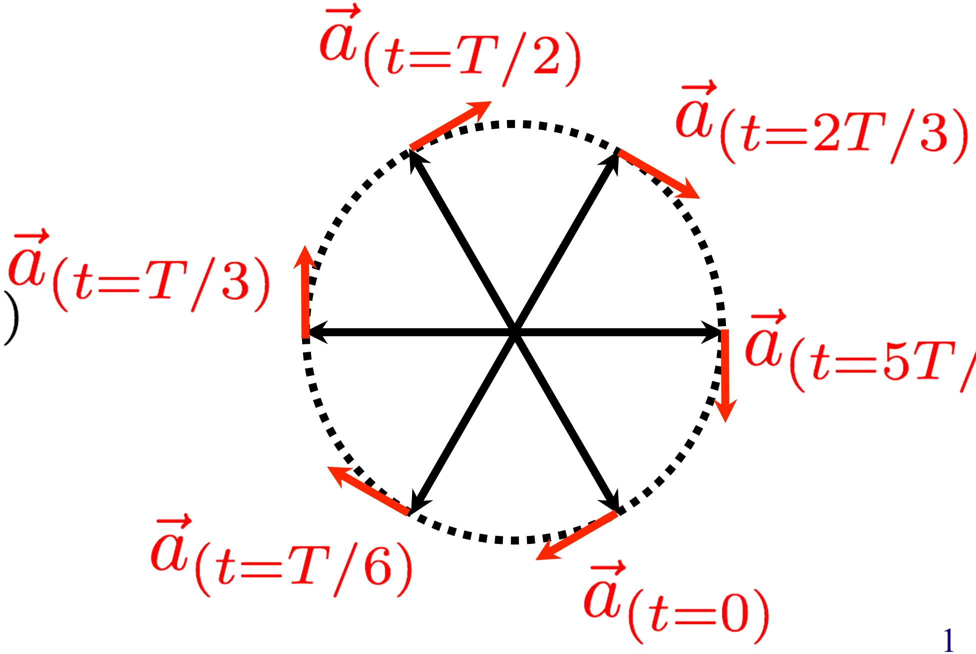
Radial Acceleration

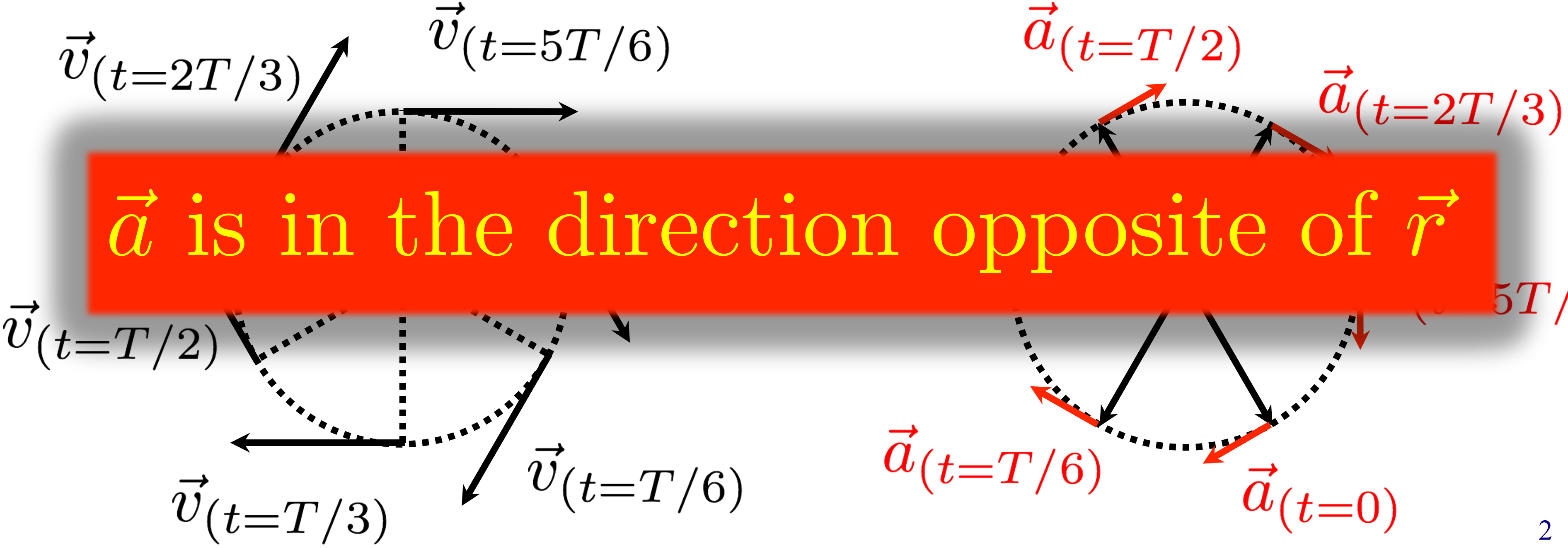
- recall, the direction of the instantaneous velocity vector is tangential to the trajectory





Radial Acceleration

- recall, the direction of the instantaneous velocity vector is tangential to the trajectory



$\vec{a}$ is in the direction opposite of $\vec{r}$

Uniform Circular Motion

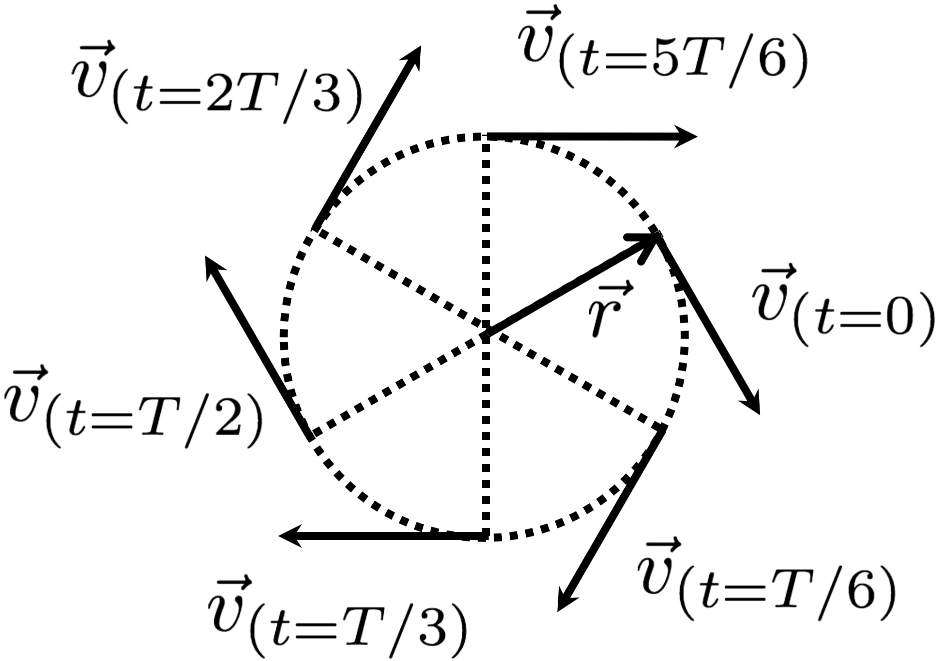
- recall the relationship between the:
- abs. value of velocity |v|
  (rate of change of the position vector)
- radius (r)
  (length of the position vector)
- angular velocity (ω)
- we can use this relationship
  to calculate radial acceleration

$$\boxed{|v| = r|\omega|}$$

Uniform Circular Motion

$$\boxed{|v| = r|\omega|}$$

- we can use this relationship
  to calculate radial acceleration

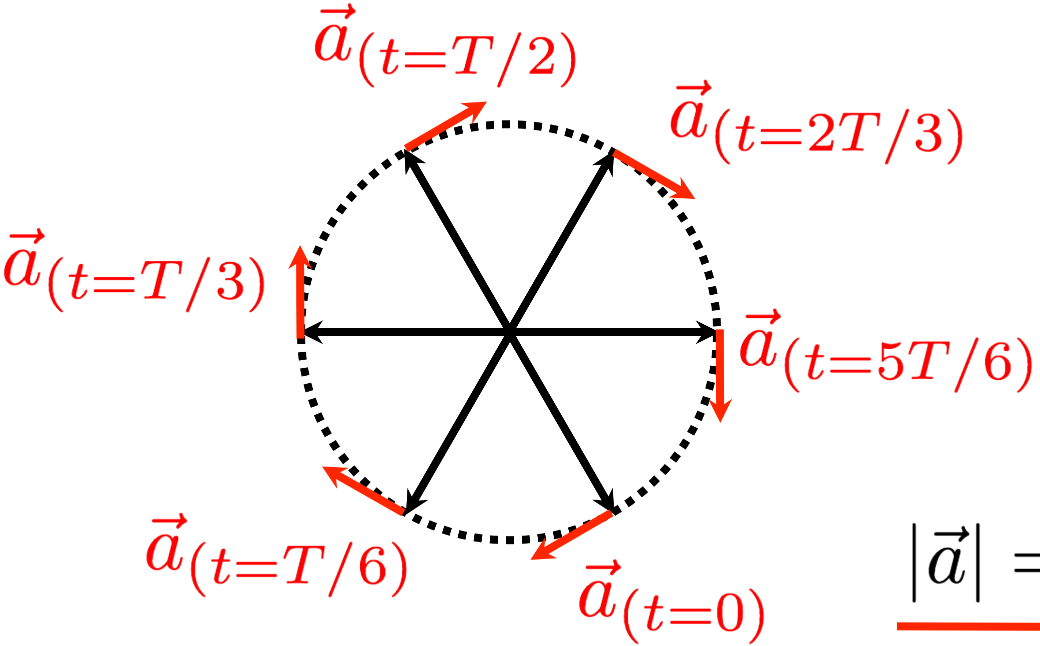


Uniform Circular Motion

We want to relate:

1) Absolute value of acceleration $|a|$
   (rate of change of the velocity vector)
2) Absolute value of velocity (v)
   (length of the velocity vector)
3) Angular velocity (ω)

Uniform Circular Motion



$$|\vec{a}| = v|\omega|$$

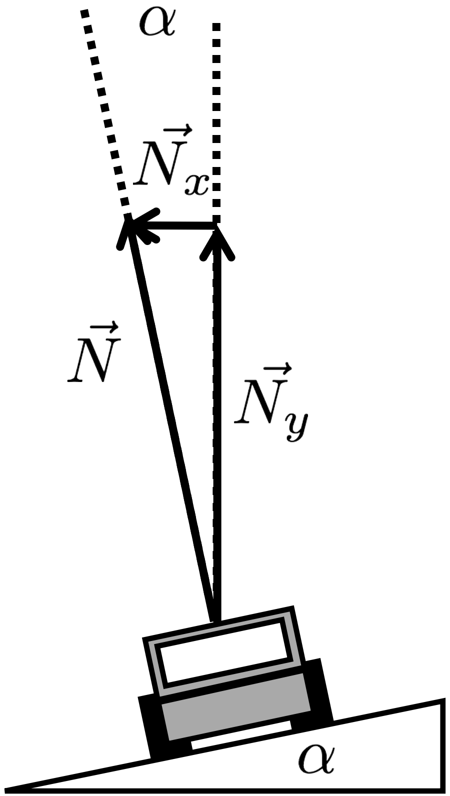
$$|\vec{v}| = r|\omega|$$

$$|\vec{a}| = r|\omega|^2$$

$$|\vec{a}| = \frac{|\vec{v}|^2}{r}$$

Banked vs Unbanked Curves

- Banking a curve at an angle produces force towards the center of the curve
- Example: a car moving at 10 m/s is entering a curve of radius 20 m. There is no friction on the road. At which angle does the bank have to be for the car not to slip in or out of the curve?
(use $g = 10\ \mathrm{m/s^2}$)



Solution

We want the horizontal component of the normal force to provide the radial acceleration, i.e.

$$a = \frac{v^2}{r}; \qquad ma = N_x$$

at the same time, the vertical component of the normal force needs to exactly balance the weight of the object

$$N_y - mg = 0 \rightarrow N_y = mg$$

The two are connected by the trigonometric equations;

$$\cancel{N_x = N\cos\alpha}$$

$$\cancel{N_y = N s}$$

$$\left.\begin{aligned} N_x &= N\sin\alpha \\ N_y &= N\cos\alpha \end{aligned}\right\} \tan\alpha = \frac{N_x}{N_y}$$

Solution, part 2

$$N_x = N_y \tan\alpha$$

$N_x = ma$; $a = \frac{v^2}{r}$

$N_y = mg$

$$N_x = m\frac{v^2}{r} = N_y \tan\alpha = mg\tan\alpha$$

$$\frac{\cancel{m}v^2}{r} = \cancel{m}g\tan\alpha$$

v and r are provided by the problem: $v = 10\,\text{m/s}$, $r = 20\,\text{m}$; $g = 10\,\text{m/s}^2$

$$\tan\alpha = \frac{v^2}{gr} = \frac{100\,\text{m}^2/\text{s}^2}{10\,\text{m/s}^2 \cdot 20\,\text{m}} = \frac{100\,\text{m}^2/\text{s}^2}{200\,\text{m}^2/\text{s}^2} = \frac{1}{2}$$

$$\alpha = \tan^{-1}(1/2) = 26.6^\circ$$

Lecture 10:
More on circular motion

Circular orbits of planets and satellites

- stabile orbits of planets and satellites: gravitational force provides radial acceleration for the system
- Newton's Law of gravity:

$$F_G = G\frac{Mm}{r^2}; \quad \text{typically } [M \gg m]$$

- radial force required for a stabile orbit:

$$F_r = ma_r = m\frac{v^2}{r} \qquad v = \sqrt{\frac{GM}{r}}$$

Orbital speed example

Compute the velocity of a satellite flying in a stable orbit just above the surface of the Moon. Take that the mass of the Moon is 7.35×10^{22} kg, its radius is 1740 km, and $G = 6.67\times10^{-11}\ m^3kg^{-1}s^{-2}$. What would be the period of this orbit?

Orbital speed example

$$\frac{\cancel{m} v^2}{\cancel{r}} = G\frac{\cancel{m} M}{r^{\cancel{2}}} \rightarrow v^2 = \frac{GM}{r}, \quad v = \sqrt{\frac{GM}{r}}$$

$$v = \sqrt{\frac{6.67 \times 10^{-11} \times 7.35 \times 10^{22}}{1740 \times 10^3}} \frac{m}{s} = 1.68 \times 10^3 \frac{m}{s}$$

$$v = 1.68 \frac{km}{s}$$ [comparable to 1 mile per second]

$$v = \frac{2\pi R}{T} \rightarrow T = \frac{2\pi R}{v} = \frac{2\pi \times 1740\ km}{1.68\ km/sec}$$

$$\underline{T = 6.51 \times 10^3\ sec\ [\approx 1.81\ hrs]}\,\|$$

Periods of planetary motion

$$F_r = F_G \quad \Rightarrow \quad G\frac{Mm}{r^2} = m\frac{v^2}{r}$$

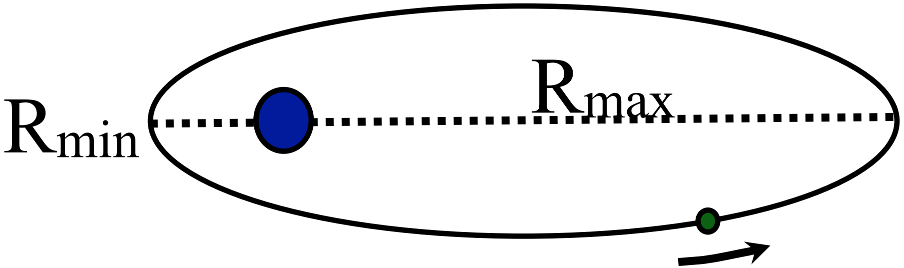
$$\boxed{G\frac{M}{r^3}} = \frac{v^2}{r^2} = \left(\frac{v}{r}\right)^2 = \boxed{\omega^2} = \left(\frac{2\pi}{T}\right)^2$$

$$\boxed{\frac{GM}{r^3}} = \frac{4\pi^2}{T^2} \quad \Rightarrow \quad \frac{GM}{4\pi^2} = \boxed{\frac{r^3}{T^2}}$$

Kepler's Laws

- three empirical laws which describe planetary motion with impressive accuracy

1) The planets travel in elliptical orbits with the Sun at one focus of the ellipse
2) A line drawn from a planet to the Sun sweeps out equal areas in equal time intervals



Kepler's Laws - continued

3) The square of the orbital period is proportional to the cube of the average distance from the planet to the Sun

this ratio is the same for all planets in the system

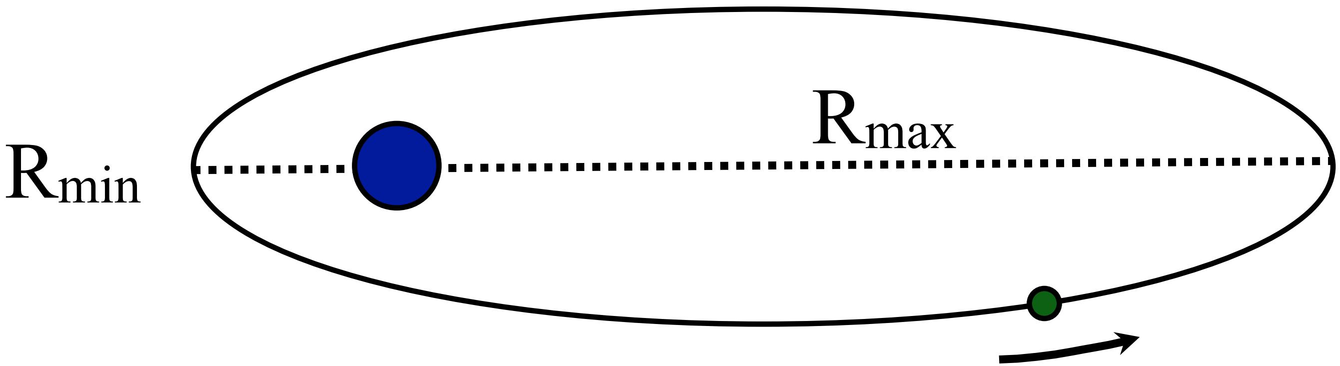
$$\boxed{\frac{T^2}{r^3} = const.}$$

$$\boxed{\left(\frac{T_2}{T_1}\right)^2 = \left(\frac{r_2}{r_1}\right)^3}$$

Example for Kepler's 2nd Law

The trajectory of planet X around its sun is a very elongated ellipse. The maximum distance of the planet from the sun is five time greater than the distance of closest approach between the planet and its sun. The (linear) speed of the planet at the point of closest approach is 30 km/s. What is the (linear) speed of the planet at the point when it is farthest from the sun?

Example for Kepler's 2nd Law



Solution

$$A \simeq \frac{1}{2} R v \Delta t \quad \text{for} \quad \Delta t \ll T$$

$$A_1 = A_2$$

$$\cancel{\tfrac{1}{2}} R_1 v_1 \cancel{\Delta t} = \cancel{\tfrac{1}{2}} R_2 v_2 \cancel{\Delta t}$$

Solution, Part 2

$$R_1 v_1 = R_2 v_2 \rightarrow v_2 = \frac{R_1}{R_2} v_1$$

$$v_1 = 30 \text{ km/s}$$

$$R_2 = 5 \times R_1 \rightarrow \frac{R_2}{R_1} = 5$$

$$v_2 = \frac{1}{5} \times 30 \text{ km/s}$$

$$v_2 = 6 \text{ km/s}$$

Example for Kepler's 3rd Law

The distance from the Sun to the Earth is called an astronomical unit of distance (AU). If Jupiter is 5.2 AU away from the Sun, and Neptune is 30.1 AU away from the Sun, what are the orbital periods of Jupiter and Neptune [in years]? Knowing the radius and the period of Earth's orbit around the Sun, and that the Moon is 0.0026 AU from the Earth, can we compute the orbital period of the Moon around the Earth?

Solution: Jupiter

$$\left(\frac{T_2}{T_1}\right)^2 = \left(\frac{R_2}{R_1}\right)^3$$

$$\text{Jupiter}: \quad \frac{R_J}{R_E} = 5.2 \rightarrow \left(\frac{T_J}{1\ \text{year}}\right)^2 = 5.2^3$$

$$\frac{T_J}{\text{year}} = 11.86 \qquad \underline{T_J = 11.86\ \text{year}}$$

Solution: Neptune

Neptune: $\frac{R_N}{R_E} = 30.1 \rightarrow \left(\frac{T_N}{1\text{ year}}\right)^2 = (30.1)^3$

$$\frac{T_N}{1\text{ year}} = 165.1 \rightarrow T_N = \underline{165.1\text{ year}}$$

Example: Orbits around the Earth

A geostationary satellite turns at the same angular velocity as the Earth. The radius of the Earth is 6,300 km. The geostationary satellite is flying at an altitude of 35,700 km. If the altitude of the moon is 384,400 km, how long does it take the moon to complete a revolution around the Earth?

Example: Orbits around the Earth

$$R_E = 6300 \text{ km}$$

$$R_{GS} = R_E + h_{GS} = [6300 + 35700] \text{ km} = 42000 \text{ km}$$

$$T_{GS} = 1 \text{ day}$$

$$R_{MOON} = R_E + h_{MOON} = [6300 + 384400] \text{ km} = 390700 \text{ km}$$

$$\left(\frac{T_{MOON}}{T_{GS}}\right)^2 = \left(\frac{R_{MOON}}{R_{GS}}\right)^3 = 28.37 \text{ days}$$

H-ITT: Sea-level orbit

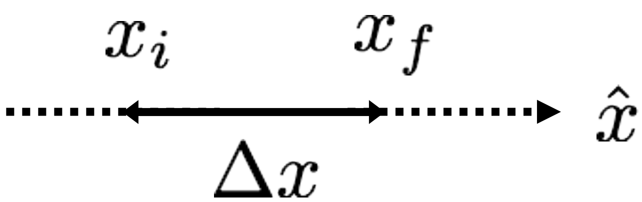
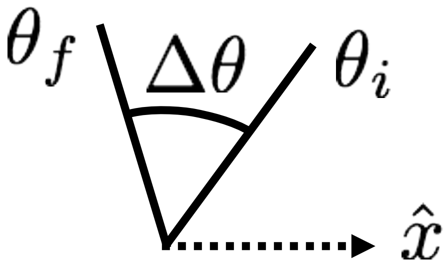
Nonuniform Circular Motion

- In all strictness, we have derived the expression for radial acceleration for uniform circular motion
- Th… un… the trajectory is constant.

These hold as long as r is constant, regardless of what happens to v_{inst} and ω_{inst}

$$\boxed{|a_r| = \frac{v_{inst}^2}{r}}$$

$$\boxed{|a_r| = r\omega_{inst}^2}$$

	Linear	Angular
		
Displacement	$\Delta x = x_f - x_i$	$\Delta\theta = \theta_f - \theta_i$
Avg. Velocity	$v_{avg} = \frac{\Delta x}{\Delta t}$	$\omega_{avg} = \frac{\Delta\theta}{\Delta t}$
Inst. Velocity	$v_{inst} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$	$\omega_{inst} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$

	Linear	Angular
Displacement	$\Delta x = x_f - x_i$	$\Delta\theta = \theta_f - \theta_i$
Avg. Velocity	$v_{avg} = \frac{\Delta x}{\Delta t}$	$\omega_{avg} = \frac{\Delta\theta}{\Delta t}$
Inst. Velocity	$v_{inst} = \lim_{\Delta t\to 0} \frac{\Delta x}{\Delta t}$	$\omega_{inst} = \lim_{\Delta t\to 0} \frac{\Delta\theta}{\Delta t}$
Average Acceleration	$a_{avg} = \frac{\Delta v}{\Delta t}$	$\alpha_{avg} = \frac{\Delta\omega}{\Delta t}$
Instantaneous Acceleration	$a_{inst} = \lim_{\Delta t\to 0} \frac{\Delta v}{\Delta t}$	$\alpha_{inst} = \lim_{\Delta t\to 0} \frac{\Delta\omega}{\Delta t}$

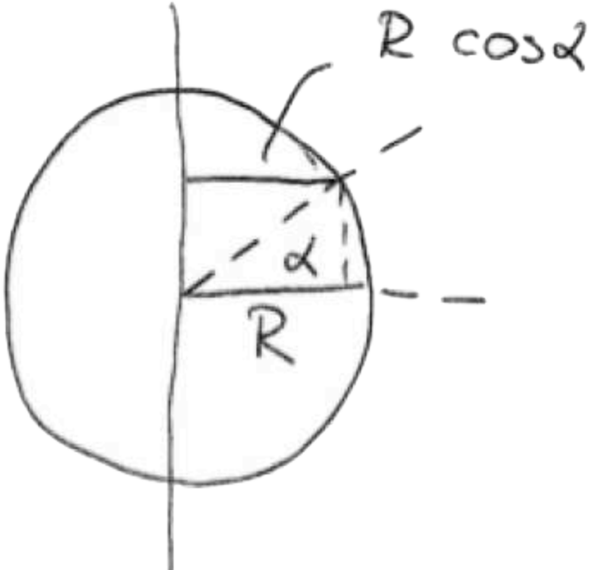
Constant Angular Acceleration vs Constant Linear Acceleration

linear, a is constant	angular, α is constant
$\Delta v = v_f - v_i = a\Delta t$	$\Delta\omega = \omega_f - \omega_i = \alpha\Delta t$
$\Delta x = v_{i,x}\Delta t + \frac{1}{2}a\Delta t^2$	$\Delta\theta = \omega_i\Delta t + \frac{1}{2}\alpha\Delta t^2$
$\Delta x = \frac{1}{2}\left(v_{i,x} + v_{f,x}\right)\Delta t$	$\Delta\theta = \frac{1}{2}\left(\omega_i + \omega_f\right)\Delta t$
$v_{f,x}^2 - v_{i,x}^2 = 2a_x\Delta x$	$\omega_f^2 - \omega_i^2 = 2\alpha\Delta\omega$

Apparent Weight and Artificial Gravity

- Application of radial force(s) to modify or create gravity as desired
- Example: a space station is shaped like a ring and rotates to simulate gravity. If the radius of the ring is 120 m, at what frequency must it rotate in order to simulate the pull of Earth's gravitational field?

Solution



$$a_r = \omega^2 \cdot (R\cos\alpha)$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{24 \times 60 \times 60 \text{ sec}}$$

$$\omega = 1{,}157 \times 10^{-5} \frac{1}{\text{sec}}$$

Solution

$$a_r = \omega^2 \times R \times \cos\alpha = \underbrace{0.0333}_{} \times \cos\alpha \ \frac{m}{s^2}$$

ω^2: $(1.157 \times 10^{-5})^2$ $\quad$ R: 6300 km

compare to 10 m/s^2

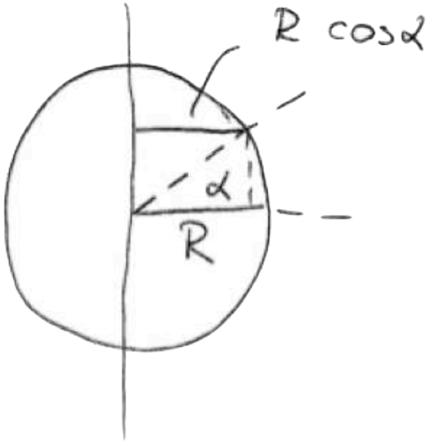
→ 0.3% effect

for $\alpha = 60°$ $\quad \cos\alpha = \frac{1}{2} \rightarrow a_r = 0.0166 \ m/s^2$

"g" Changes With Latitude

What is the radial acceleration due to Earth's rotation for an object in Helsinki, Finland, which is located at a latitude of 60° with respect to the equator?

"g" Changes With Latitude



$$a_r = \omega^2 \cdot (R\cos\alpha)$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{24\times 60\times 60\ \text{sec}}$$

$$\omega = 1.157\times 10^{-5}\ \frac{1}{\text{sec}}$$

$$a_r = \underbrace{\omega^2}_{(1.157\times 10^{-5})^2} \times \underbrace{R}_{6300\,\text{km}} \times \cos\alpha = \underbrace{0.0333}_{} \times \cos\alpha\ \frac{m}{s^2}$$

compare to $10\ m/s^2$
→ 0.3% effect

for $\alpha = 60^\circ$ $\cos\alpha = \frac{1}{2}$ → $a_r = 0.0166\ m/s^2$