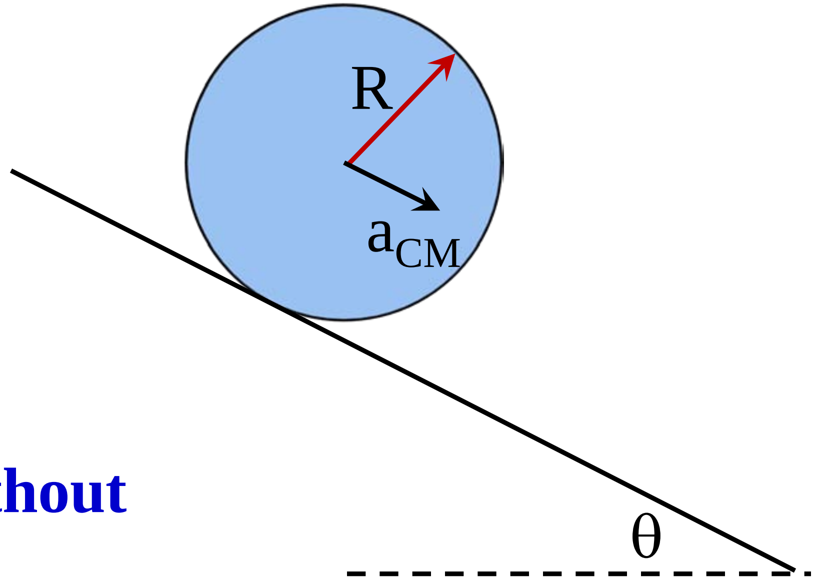
Rotational Motion (cont.)

Linear acceleration of rolling objects

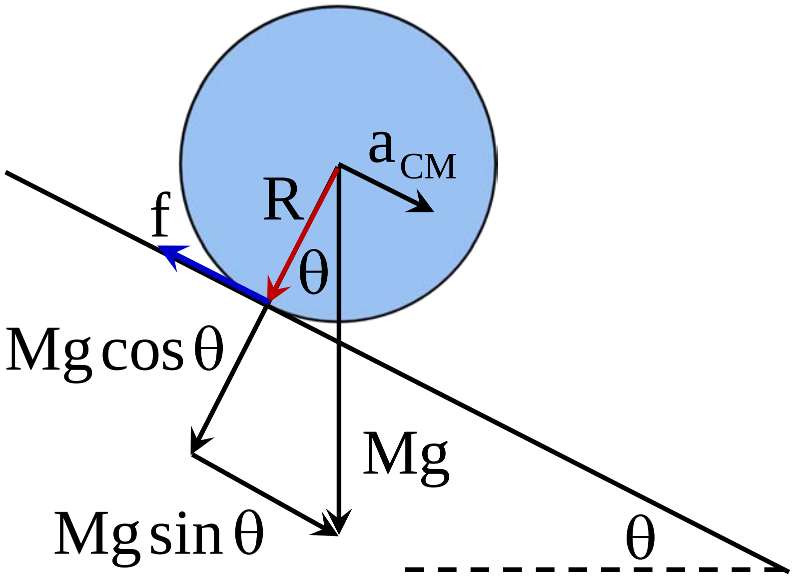
Consider a **round object** (this could be a cylinder, hoop, sphere or spherical shell) having mass M, radius R and rotational inertia I about its center of mass, **rolling without slipping** down an inclined plane.



What is the **linear acceleration** of the object's center of mass, a_{CM}, down the incline?

We analyze this as follows:

The force of **gravity**, Mg, acting straight down is resolved into components **parallel** and **perpendicular** to the **incline**.



Since the object rolls without slipping there is a force of **friction**, f, acting on the object, at it's point of contact with the incline, in the direction **up the incline**.

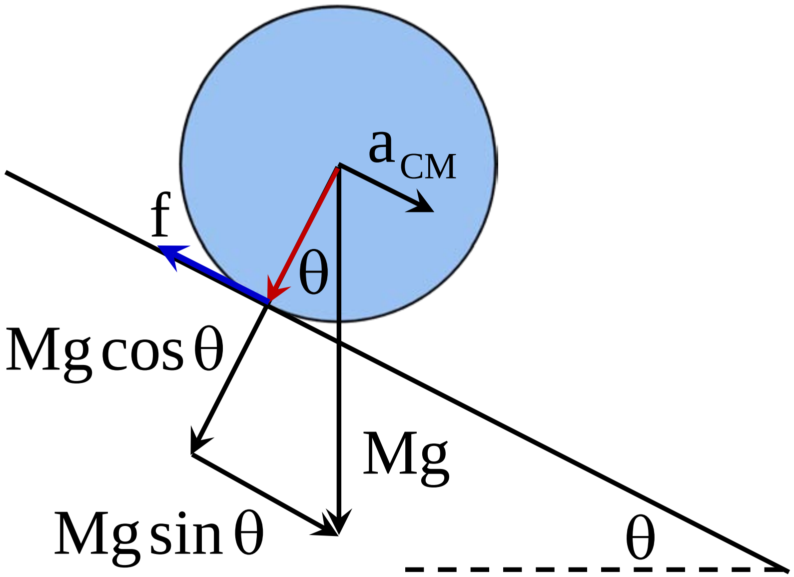
Newton's 2nd law gives then for acceleration down the incline,

$$\sum F_{||} = Ma_{CM}$$

$$Mg\sin\theta - f = Ma_{CM}$$

The force of friction also causes a **torque** around the center of mass having **lever arm** R so we can also write,

$$\tau = Rf = I\alpha$$



Solving for the friction,

$$f = \frac{I}{R}\alpha$$

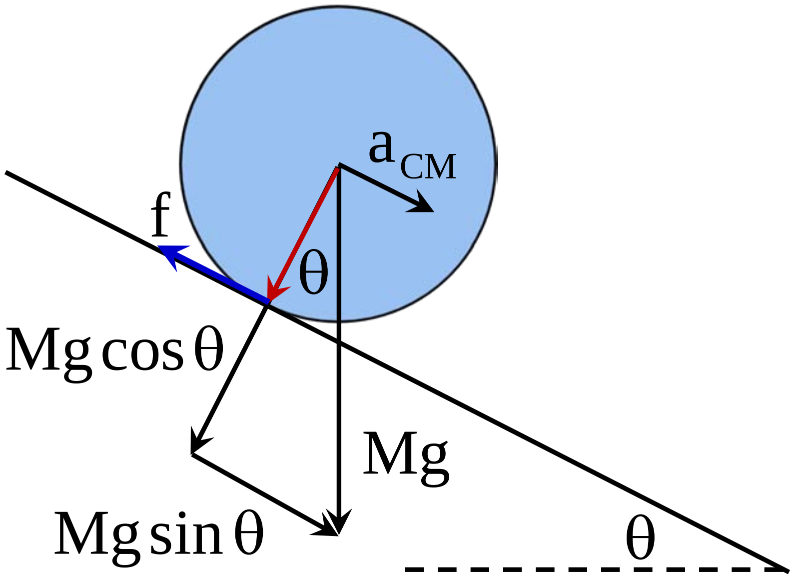
This is used in the expression derived from the 2nd law:

$$Mg\sin\theta - f = Ma_{CM}$$

$$Mg\sin\theta - \frac{I}{R}\alpha = Ma_{CM}$$

The objects **angular acceleration** is related to the **linear acceleration** of the edge that contacts the incline by,

$$a = R\alpha$$



Since the object rolls without slipping this has the same magnitude as a_{CM} so we have that,

$$\alpha = \frac{a_{CM}}{R}$$

Using this in,

$$Mg\sin\theta - \frac{I}{R}\alpha = Ma_{CM}$$

$$Mg\sin\theta - \frac{I}{R}\frac{a_{CM}}{R} = Ma_{CM} \quad \rightarrow \quad -Ma_{CM} - \frac{I}{R^2}a_{CM} = -Mg\sin\theta$$

Multiplying through by –1,

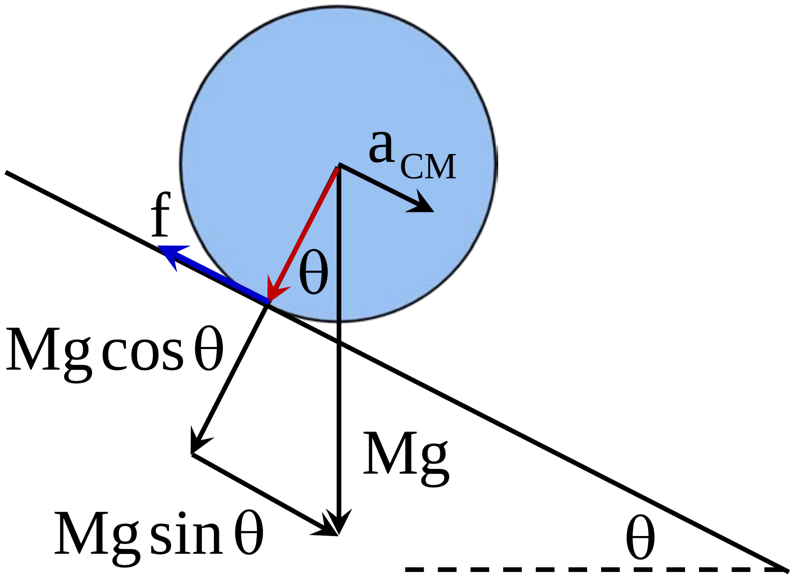
$$Ma_{CM} + \frac{I}{R^2} a_{CM} = Mg\sin\theta$$

$$Ma_{CM}\left(1 + \frac{I}{MR^2}\right) = Mg\sin\theta$$

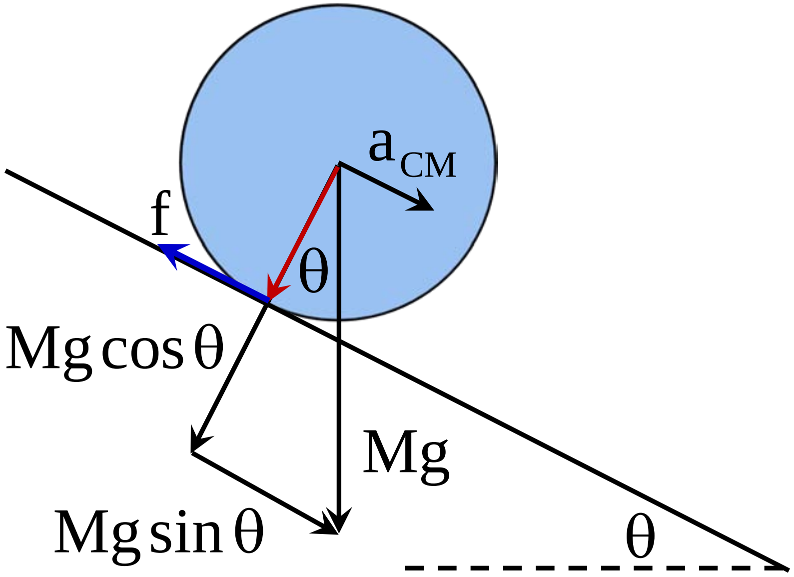
$$a_{CM}\left(1 + \frac{I}{MR^2}\right) = g\sin\theta$$

So that ,finally,

$$a_{CM} = \frac{g\sin\theta}{1 + \frac{I}{MR^2}}$$



$$a_{CM} = \frac{g\sin\theta}{1 + \frac{I}{MR^2}}$$



If the object is a solid cylinder, $I = \frac{1}{2}MR^2$

$$a_{CM} = \frac{g\sin\theta}{1 + \frac{\frac{1}{2}MR^2}{MR^2}} = \frac{g\sin\theta}{1 + \frac{1}{2}} = \frac{g\sin\theta}{\frac{3}{2}} = \frac{2}{3}g\sin\theta \quad \text{(solid cylinder)}$$

If instead the object is cylindrical shell, with all its mass at the rim $I = MR^2$ and,

$$a_{CM} = \frac{g\sin\theta}{1 + \frac{MR^2}{MR^2}} = \frac{g\sin\theta}{1+1} = \frac{1}{2}g\sin\theta \quad \text{(cylindrical shell)}$$

Angular Momentum

The rotational analog to linear momentum ($p = mv$) is angular momentum,

$$L = I\omega$$

Recall that linear momentum is important because given a system of objects, in the absence of external forces, no matter how the objects of the system interact with each other, their total linear momentum is conserved. I.e. with,

$$\vec{P} = \sum_{i=1}^{n} \vec{p}_i$$

For any two times,

$$\vec{P}_f = \vec{P}_i$$

Thus if there are **two objects** with linear momenta $\vec{p}_1$ and $\vec{p}_2$ so that their total momentum is at one time,

$$\vec{P}_i = \vec{p}_{1i} + \vec{p}_{2i}$$

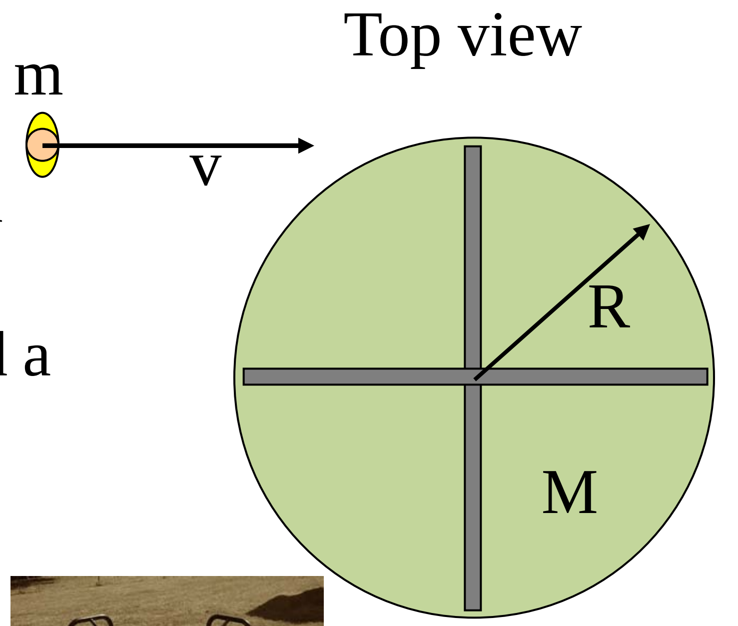
Then no matter how they **interact, collide, attract or repel** each other, in the absence of external forces, this total momentum will **not change** so we have that,

$$\vec{p}_{1f} + \vec{p}_{2f} = \vec{p}_{1i} + \vec{p}_{2i}$$

Similarly, for rotational motion, in the absence of external torques (i.e. $\sum \tau = 0$), the **angular momentum is conserved** meaning that for any two times,

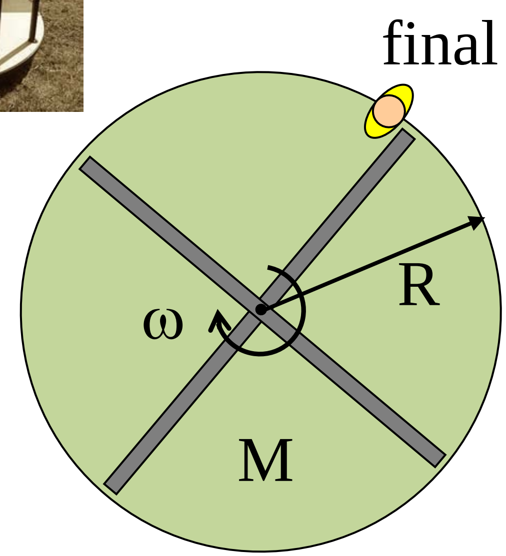
$$L_f = L_i$$

<u>Example</u>



A boy (mass m) runs with speed v and jumps onto the edge of an initially stationary merry-go-round (also called a carousel).

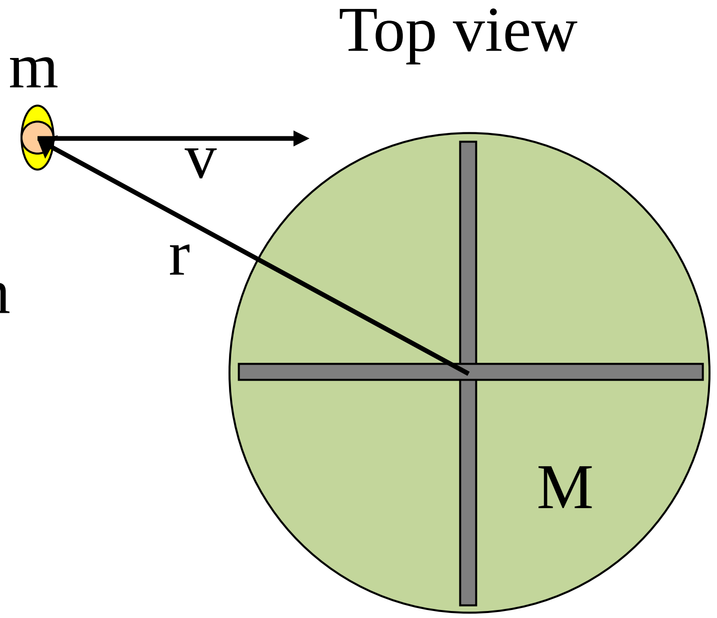
What is the angular velocity of the carousel (and the boy) after he has jumped on?



Conservation of angular momentum requires,

$$L_f = L_i$$

We must first consider the boy's **angular momentum** about the rotation axis of the carousel when he is running in a **straight line**. His angular momentum is,



$$\ell_B = I_B \omega_B$$

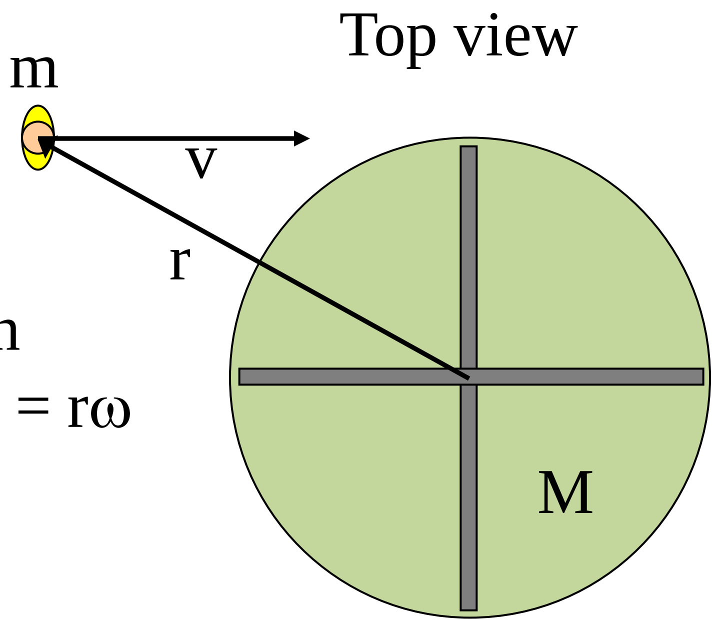
Where I_B is his rotational inertia about the axis and ω_B is his angular velocity about the axis.

We treat the boy as a point object of mass m, making his rotational inertia about the carousel axis

$$I_B = mr^2$$

Where r is his (changing) distance to the carousel axis.

What is the boy's angular velocity ω_B about the axis?



When we previously considered motion on a **circular trajectory** we had that $v_t = r\omega$ where the subscript t reminds us that this was velocity tangent to the circle.

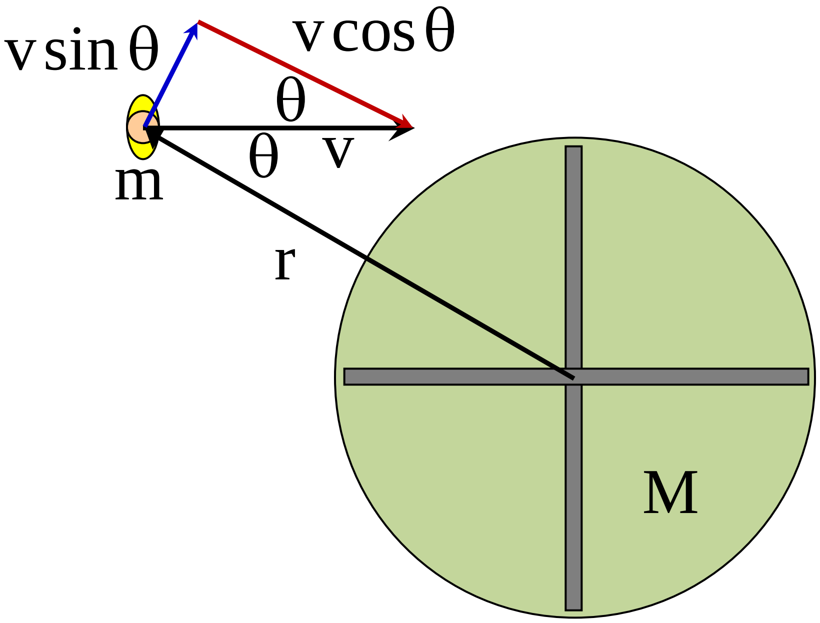
Tangent to the circle means the **component of the velocity** that is **perpendicular** to a **line drawn to the center of the circle** (all of v, when the trajectory is circular).

For this more general case we must **decompose** the velocity vector into components **parallel** and **perpendicular** to the line to the rotation axis.

Then,

$$v\sin\theta = r\omega_B$$

or, $$\omega_B = \frac{v}{r}\sin\theta$$



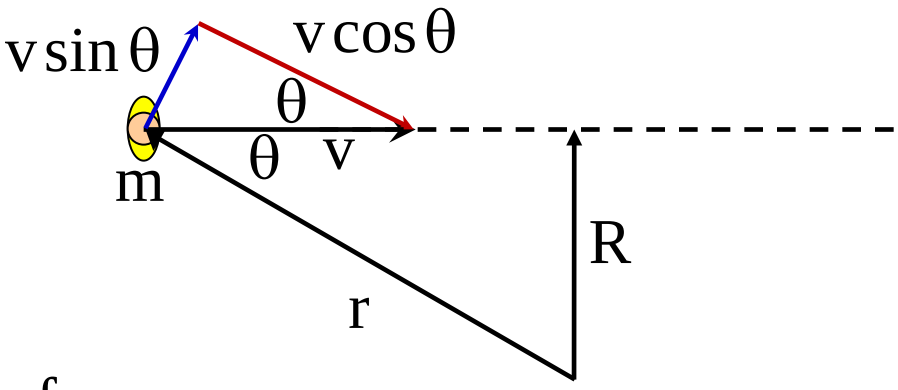
Using this and $I_B = mr^2$

in,

$$\ell_B = (mr^2)\frac{v}{r}\sin\theta$$

gives,

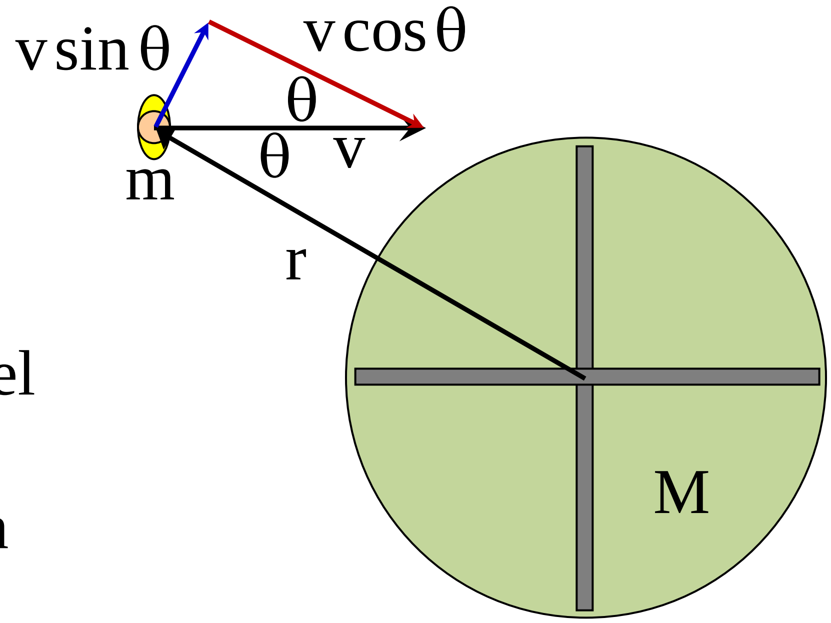
$$\ell_B = mrv\sin\theta = mv(r\sin\theta)$$



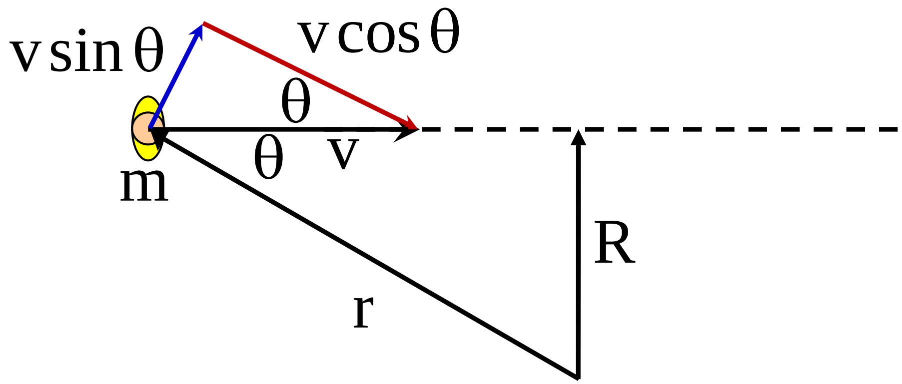
But notice that along his linear trajectory $r\sin\theta$ is also the point of closest approach to the axis which here is R so that,

$$\ell_B = mv(r\sin\theta) = mvR$$

Since all of these m, v and R are **constant** what we have just shown is that despite the fact that r and θ are both changing as the boy approaches the carousel his angular momentum about the axis remains constant and is given by



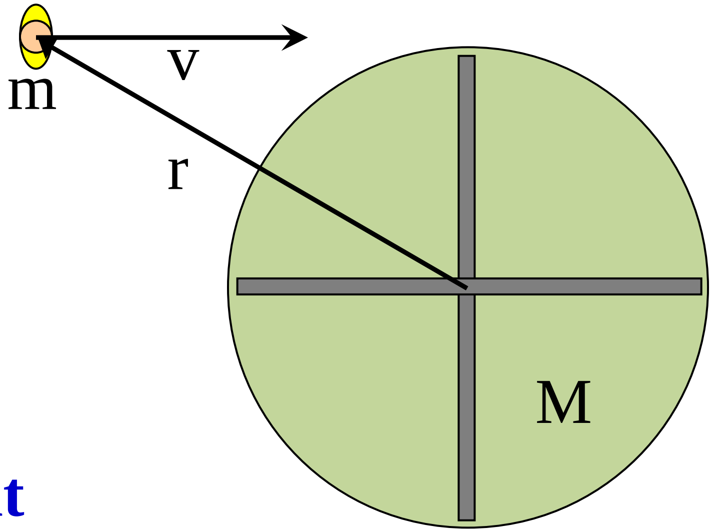
$$\ell_B = mvR$$



Where R is the point of closest approach to the axis.

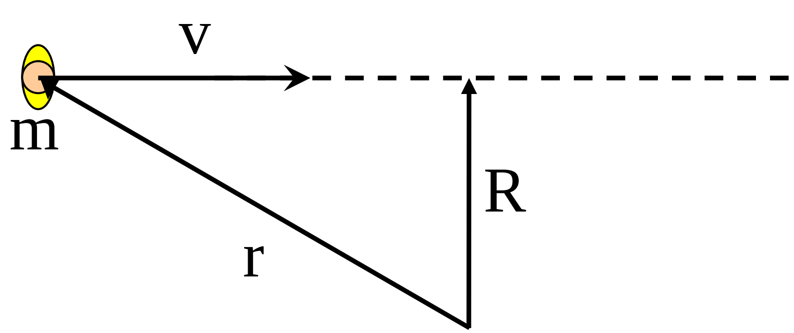
(this must be true for angular momentum to be conserved. I.e. ℓ_B calculated at any two times along this trajectory had better be the same).

Having shown this once, we don’t need to re-derive it every time it comes up.



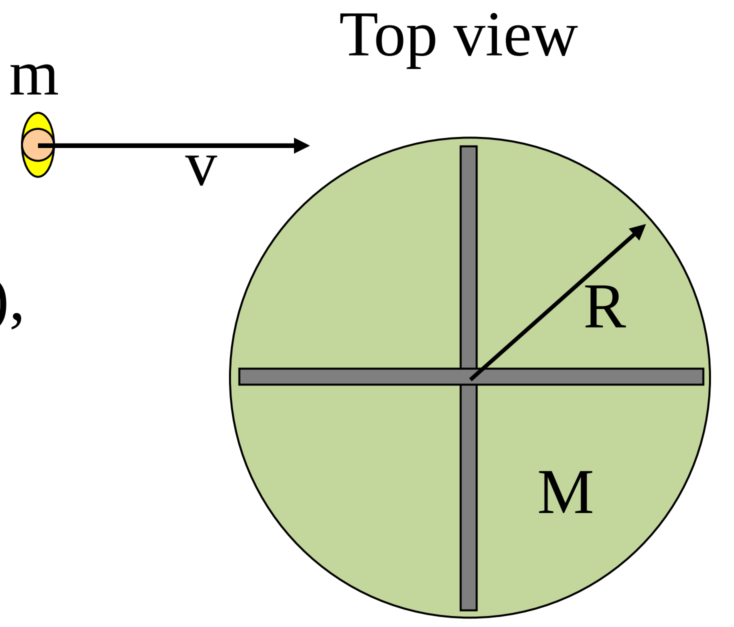
What needs to be remembered is that an object travelling **linearly**, with **constant v**, on a trajectory to pass an **arbitrary point** has a **constant angular momentum** about that point given by,

$$\ell_B = mvR$$



Where R is the point of **closest approach** of the trajectory to the point.

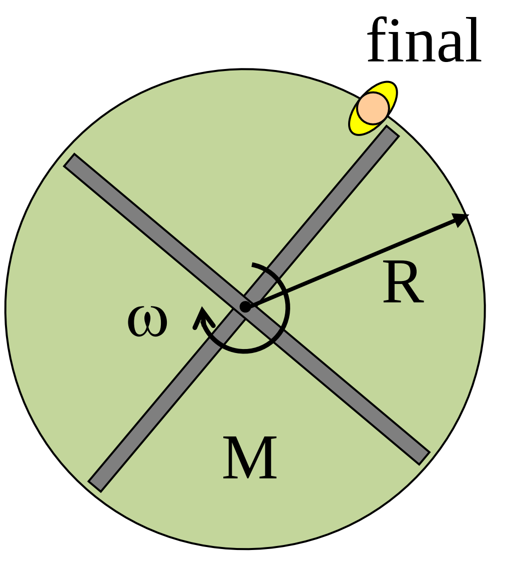
The initial angular momentum of the system is the sum of the initial angular momenta of the boy $\ell_{Bi} = mvR$ and the carousel $\ell_{Ci} = 0$ (since $\omega_{Ci} = 0$),



$$L_i = \ell_{Bi} + \ell_{Ci} = mvR$$

After the boy jumps on we have

$$L_f = I_f \omega_f$$



Where I_f is the combined rotational inertia of the boy and the carousel. The carousel can be treated as a uniform disk with

$$I_C = \frac{1}{2} MR^2$$

For rotation about the same axis rotational inertia are simply additive so,

$$I_f = \frac{1}{2}MR^2 + mR^2 = \left(\frac{M}{2} + m\right)R^2$$

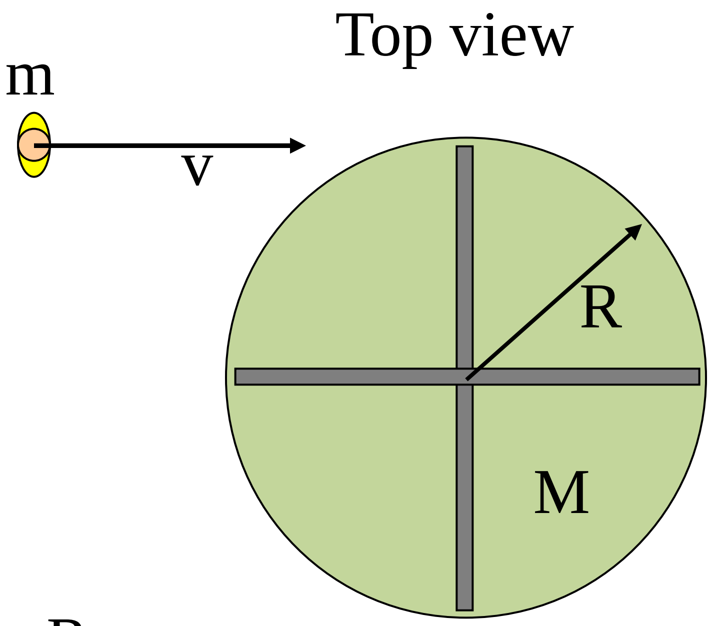
Then,

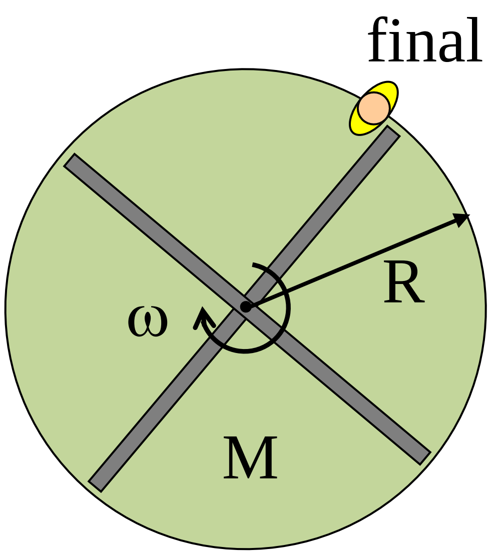
$$L_f = I_f\omega_f = \left(\frac{M}{2} + m\right)R^2\omega_f = L_i = mvR$$

Solving for ω_f ,

$$\omega_f = \frac{mv}{\left(\frac{M}{2} + m\right)R}$$

Check behavior if $M \to 0$.

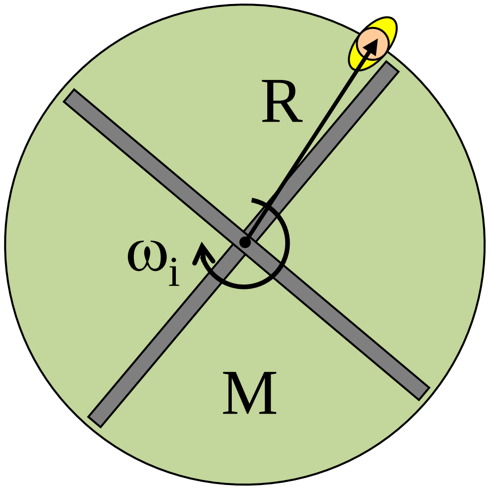




New example

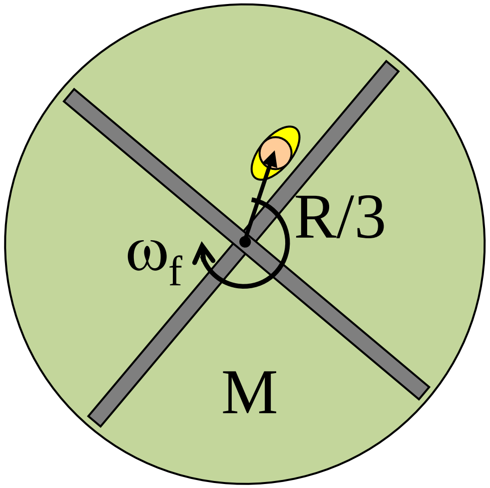
Suppose the boy is on the **rim** of carousel at radius R from the center and the two have **angular velocity ω_i**. The boy then pulls himself along the hand rail until he is a distance R/3 from the center.

initial



What is ω_f after this move?

final



$$L_f = L_i$$

$$\ell_{Bf} + \ell_{Cf} = \ell_{Bi} + \ell_{Ci}$$

$$m\left(\frac{R}{3}\right)^2 \omega_f + \frac{1}{2}MR^2\omega_f = mR^2\omega_i + \frac{1}{2}MR^2\omega_i$$

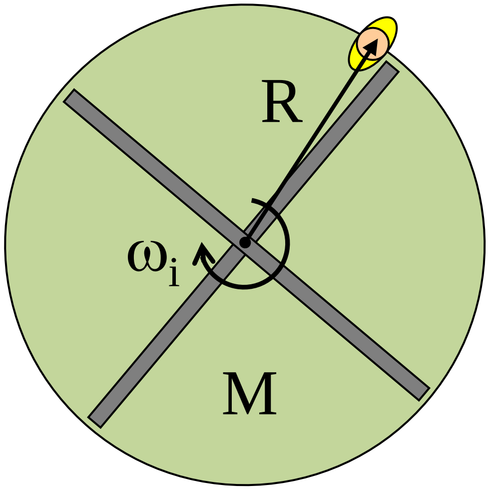
$$m\left(\frac{1}{3}\right)^2 R^2\omega_f + \frac{1}{2}MR^2\omega_f = mR^2\omega_i + \frac{1}{2}MR^2\omega_i$$

$$\left(\frac{m}{9}+\frac{M}{2}\right)R^2\omega_f = \left(m+\frac{M}{2}\right)R^2\omega_i$$

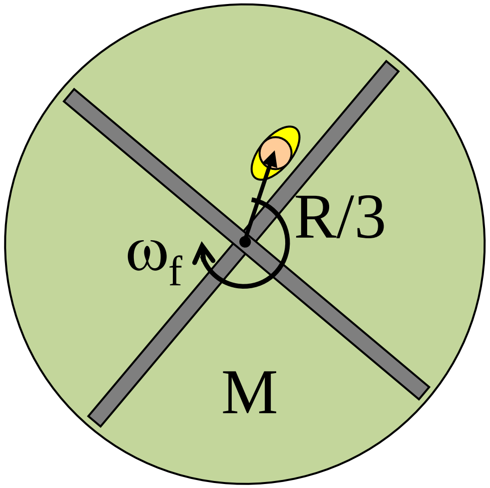
$$\left(\frac{m}{9}+\frac{M}{2}\right)\omega_f = \left(m+\frac{M}{2}\right)\omega_i$$

$$\omega_f = \frac{\left(m+\frac{M}{2}\right)}{\left(\frac{m}{9}+\frac{M}{2}\right)}\omega_i$$

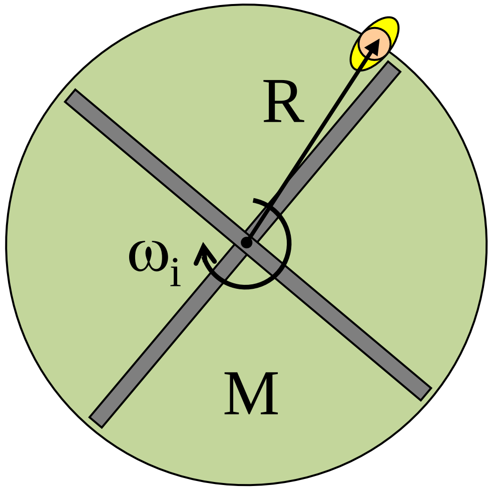
initial



final

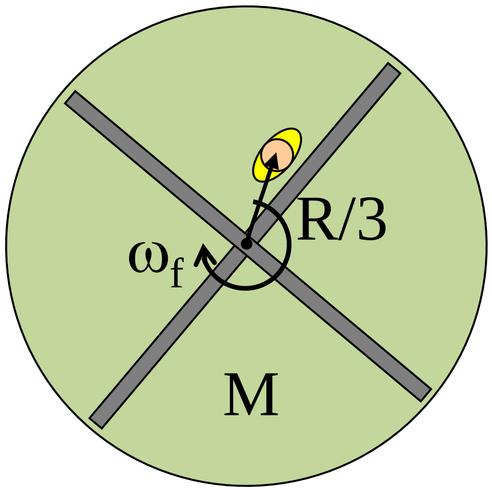


$$\omega_f = \frac{\left(m + \frac{M}{2}\right)}{\left(\frac{m}{9} + \frac{M}{2}\right)} \omega_i$$



Suppose that M = 6m then,

$$\omega_f = \frac{\left(m + \frac{6m}{2}\right)}{\left(\frac{m}{9} + \frac{6m}{2}\right)} \omega_i$$

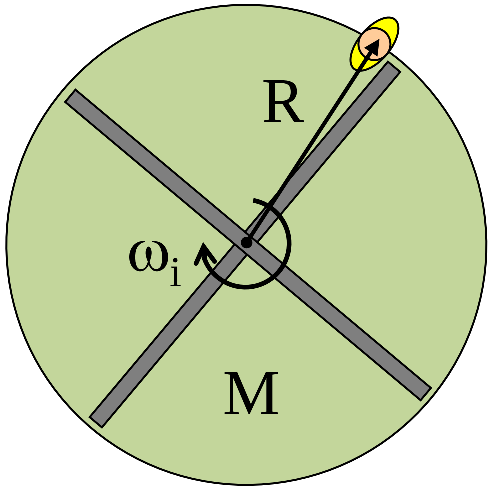


$$\omega_f = \frac{(1+3)}{\left(\frac{1}{9} + 3\right)} \omega_i = 1.29\omega_i$$

So the angular velocity increases when the boy moves in.

If the boy's mass is m = 25 kg, the carousel's mass is 6m = 150 kg the initial angular velocity was ω_i = 2 rad/s and R = 1.5 m. How much work did the boy do in pulling himself in to R/3?

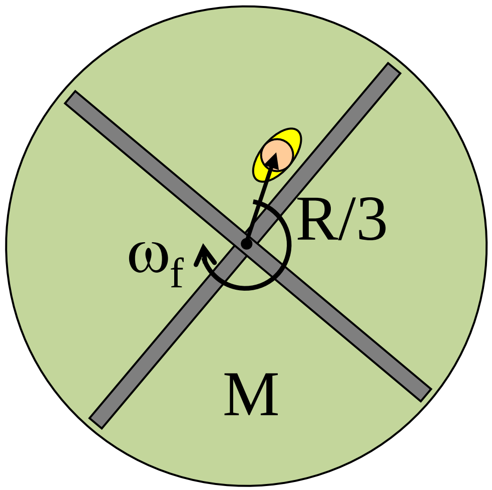
initial



$$W_{nc} = \Delta K + \Delta U$$

There is no ΔU here so,

final



$$W_{nc} = \Delta K = K_f - K_i$$

$$W_{nc} = \frac{1}{2} I_f \omega_f^2 - \frac{1}{2} I_i \omega_i^2$$

Above we found that,

$$\omega_f = 1.29\omega_i = 2.58 \frac{\text{rad}}{\text{s}}$$

$$I_i = mR^2 + \frac{1}{2}MR^2$$

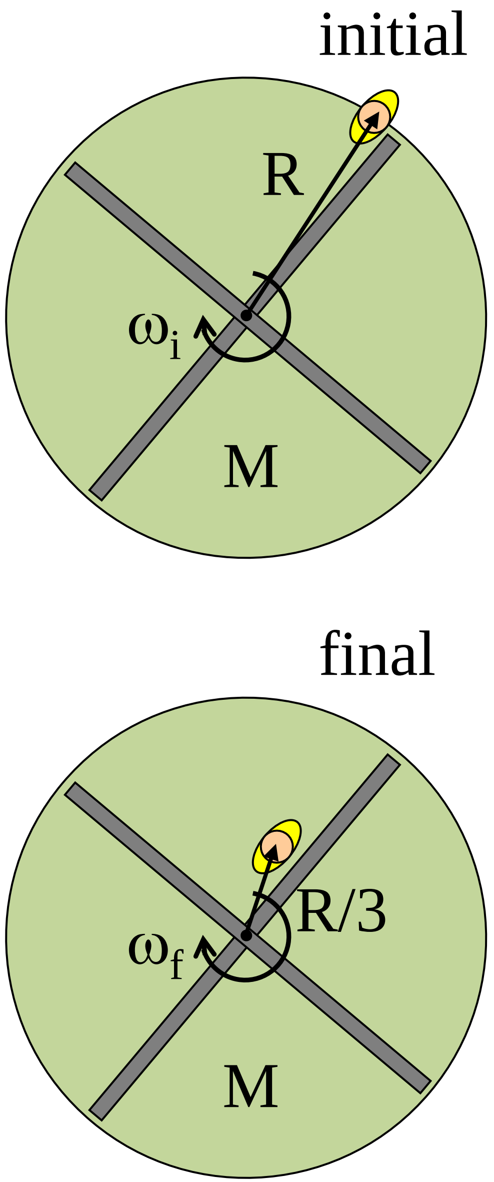
$$I_i = \left(m + \frac{1}{2}6m \right)R^2 = 4mR^2$$

$$I_i = 4(25\text{kg})(1.5\text{m})^2 = 225\text{kg}\cdot\text{m}^2$$

$$I_f = m\left(\frac{R}{3}\right)^2 + \frac{1}{2}MR^2$$

$$I_f = m\left(\frac{1}{9}\right)R^2 + \frac{1}{2}6mR^2 = 3.11mR^2$$

$$I_f = 3.11(25\text{kg})(1.5\text{m})^2 = 175\text{kg}\cdot\text{m}^2$$



So with,

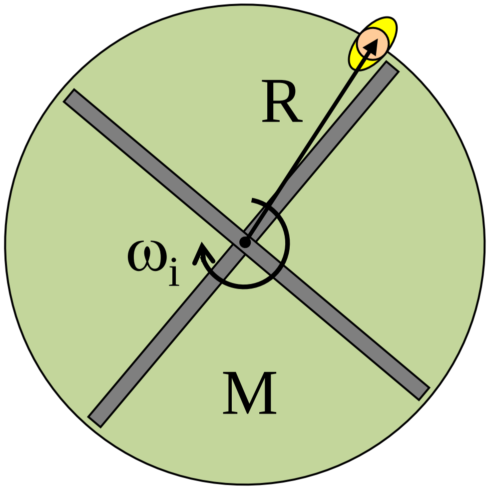
$$I_i = 225\,\text{kg}\cdot\text{m}^2 \qquad \omega_i = 2.00\,\frac{\text{rad}}{\text{s}}$$

$$I_f = 175\,\text{kg}\cdot\text{m}^2 \qquad \omega_f = 2.58\,\frac{\text{rad}}{\text{s}}$$

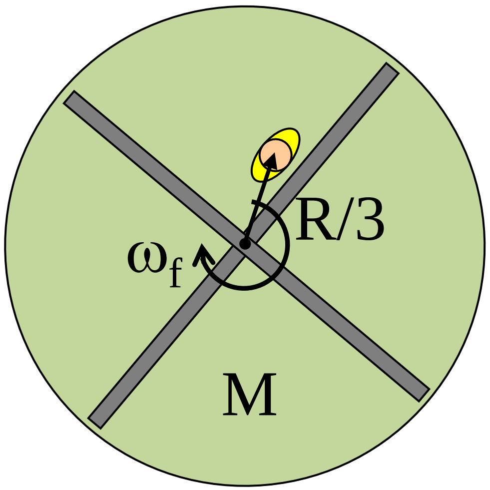
$$W_{nc} = \frac{1}{2}I_f\omega_f^2 - \frac{1}{2}I_i\omega_i^2$$

$$W_{nc} = 582.4\text{ J} - 450\text{ J} = 132\text{ J}$$

initial



final



Summary of translational quantities and their rotational analogs

Translation	Rotation
m	I
$\vec{\mathbf{F}}$	τ
$\vec{\mathbf{a}}$	α
$\sum\vec{\mathbf{F}} = m\vec{\mathbf{a}}$	$\sum\tau = I\alpha$
Δx	$\Delta\theta$
$W = F_x\Delta x$	$W = \tau\Delta\theta$
$\vec{\mathbf{v}}$	ω
$K = \frac{1}{2}mv^2$	$K = \frac{1}{2}I\omega^2$
$\vec{\mathbf{p}} = m\vec{\mathbf{v}}$	$L = I\omega$
$\sum\vec{\mathbf{F}} = \lim_{\Delta t\to 0}\frac{\Delta\vec{\mathbf{p}}}{\Delta t}$	$\sum\tau = \lim_{\Delta t\to 0}\frac{\Delta L}{\Delta t}$
If $\sum\vec{\mathbf{F}} = 0$, $\vec{\mathbf{p}}$ is conserved	If $\sum\tau = 0$, L is conserved