Circular Motion: Angular Variables

- **Arc Length:**
  The arc length \( s \) is related to the angle \( \theta \) (in radians = rad) as follows:
  \[
  s = r \theta 
  \]
  \( \theta = \frac{s}{r} \)
  
  (360° = 2\pi rad)

- **Angular Displacement and Angular Velocity:**
  \[
  \Delta \theta = \theta_f - \theta_i 
  \]
  \[
  \omega = \lim_{\Delta t \to 0} \Delta \theta \frac{d\theta}{dt} = \frac{d\theta}{dt} 
  \]
  (radians/second)

- **Tangential Velocity and Angular Velocity:**
  \[
  v_t = \left| \vec{v}_t \right| = \frac{ds}{dt} = r|\omega| 
  \]
  \[
  v_t = r\omega
  \]

- **Period and Frequency (constant \( \omega \))**:
  \[
  C = 2\pi r = v_tT 
  \]
  \[
  T = \frac{2\pi r}{v_t} = \frac{2\pi}{\omega} 
  \]
  \[
  f = \frac{1}{T} 
  \]
  (revolutions/second)
Circular Motion: Radial Acceleration

- **Radial (centripetal) Acceleration:**
  
  Centripetal acceleration = “toward the center”

  \[ \Delta \vec{v}_t = -(v_t \Delta \theta) \hat{r} \]

  \[
  \vec{a}_{radial} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}_t}{\Delta t} = -(v_t \frac{\Delta \theta}{\Delta t}) \hat{r} = -(v_t \omega) \hat{r} = -(r \omega^2) \hat{r}
  \]

  \[
  a_{radial} = |\vec{a}_{radial}| = r \omega^2 = \frac{v_t^2}{r}
  \]

  Magnitude = \((v_t)^2/r\)

  Direction = toward the center of the circle
Example Problem

• A puck of mass $m$ slides in a circle of radius $r = 0.5 \text{ m}$ on a frictionless table while attached to a hanging cylinder of mass $M = 2m$ by a cord through a hole in the table. What speed of the mass $m$ keeps the cylinder at rest?

Answer: $v = \sqrt{\frac{Mgr}{m}} \approx 3.13 m / s$

Solution:

$F_T - Mg = 0$

$F_T = ma_{radial} = m\frac{v^2}{r}$

$m\frac{v^2}{r} = Mg \quad \Rightarrow \quad v = \sqrt{\frac{Mgr}{m}} = \sqrt{\frac{2mgr}{m}} = \sqrt{2gr}$

$= \sqrt{2(9.8 m / s^2)(0.5 m)} \approx 3.13 m / s$
Example: Conical Pendulum

- A stone of mass \( m \) is connected to a cord with length \( L \) and negligible mass. The stone is undergoing uniform circular motion in the horizontal plane. If the cord makes an angle \( \phi \) with the vertical direction, what is the period of the circular motion?

**x-component:**

\[
T \sin \phi = ma_x = ma_{radial} = mr \omega^2 \quad (1)
\]

**y-component:**

\[
T \cos \phi - mg = ma_y = 0 \quad \Rightarrow \quad T \cos \phi = mg \quad (2)
\]

Divide (1) by (2)

\[
\tan \phi = \frac{r \omega^2}{g} = \frac{(L \sin \phi) \omega^2}{g} \quad \Rightarrow \quad \omega = \sqrt{\frac{g}{L \cos \phi}}
\]

\[
T_{\text{period}} = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L \cos \phi}{g}}
\]
A conical pendulum is constructed from a stone of mass $M$ connected to a cord with length $L$ and negligible mass. The stone is undergoing uniform circular motion in the horizontal plane as shown in the figure. If the cord makes an angle $\theta = 30^\circ$ with the vertical direction and the period of the circular motion is 4 s, what is the length $L$ of the cord (in meters)?

Answer: 4.59

% Right: 34%

\[ F_T \cos \phi - Mg = 0 \quad (1) \]

\[ F_T \sin \phi = Ma_x = M \frac{v^2}{R} \quad (2) \]

Divide (2) by (1)

\[ \tan \phi = \frac{v^2}{Rg} \]

\[ T = \frac{2\pi R}{v} \]

\[ \tan \phi = \frac{4\pi^2 R}{gT^2} \]

\[ R = \frac{gT^2 \tan \phi}{4\pi^2} \]

\[ \sin \phi = \frac{R}{L} \]

\[ L = \frac{gT^2}{4\pi^2 \cos \phi} \approx \frac{(9.8 \text{ m/s}^2)(4 \text{ s})^2}{4\pi^2 \cos(30^\circ)} \approx 4.59 m \]
Example Problem: Unbanked Curves

- A car of mass $M$ is traveling in a circle with radius $R$ on a flat highway with speed $v$. If the static coefficient of friction between the tires and the road is $\mu_s$, what is the maximum speed of the car such that it will not slide?

**x-component:**

$$f_s = Ma_x = Ma_{radial} = M \frac{v^2}{R}$$

**y-component:**

$$N - Mg = Ma_y = 0$$

$$f_s \leq \mu_s N$$

$$v_{max}^2 = \frac{R}{M} (f_s)_{max} = \frac{R}{M} (\mu_s N) = \frac{R}{M} (\mu_s Mg) = \mu_s gR$$

$$v_{max} = \sqrt{\mu_s gR}$$
Exam 2 Spring 2011: Problem 4

Near the surface of the Earth, a car is traveling at a constant speed \( v \) around a flat circular race track with a radius of 50 m. If the coefficients of kinetic and static friction between the car’s tires and the road are \( \mu_k = 0.1 \), \( \mu_s = 0.4 \), respectively, what is the maximum speed the car can travel without slipping?

Answer: 14 m/s
% Right: 74%

\[
f_s = Ma_x = Ma_{radial} = M \frac{v^2}{R}
\]

\[
F_N - Mg = Ma_y = 0
\]

\[
f_s \leq \mu_s F_N
\]

\[
\frac{v^2_{max}}{M} (f_s)_{max} = \frac{R}{M} (\mu_s F_N) = \frac{R}{M} (\mu_s Mg) = \mu_s gR
\]

\[
v_{max} = \sqrt{\mu_s gR} = \sqrt{(0.4)(9.8 m/s^2)(50 m)} = 14 m/s
\]
Example Problem: Banked Curves

- If the car in the previous problem is traveling on a banked road (angle $\theta$), what is the maximum speed of the car such that it will not slide?

**x-component:**

$$f_s \cos \theta + N \sin \theta = Ma_{radial} = M \frac{v^2}{R}$$

**y-component:**

$$N \cos \theta - f_s \sin \theta - Mg = Ma_y = 0$$

$$N = \frac{Mg}{\cos \theta - \mu_s \sin \theta}$$

$$v_{\text{max}}^2 = \frac{R}{M} [(f_s)_{\text{max}} \cos \theta + N \sin \theta] = \frac{R}{M} (\mu_s \cos \theta + \sin \theta)N$$

$$= Rg \frac{(\mu_s \cos \theta + \sin \theta)}{(\cos \theta - \mu_s \sin \theta)} = Rg \frac{(\mu_s + \tan \theta)}{(1 - \mu_s \tan \theta)}$$

$$v_{\text{max}} = \sqrt{\frac{Rg(\mu_s + \tan \theta)}{(1 - \mu_s \tan \theta)}}$$
Rolling Without Slipping: Rotation & Translation

If a cylinder of radius $R$ rolls without slipping along the $x$-axis then:

$$x = s = r\theta$$

$$v = \frac{dx}{dt} = R \frac{d\theta}{dt} = R\omega$$

- Translational Speed
- Rotational Speed