Circular Motion: Angular Variables

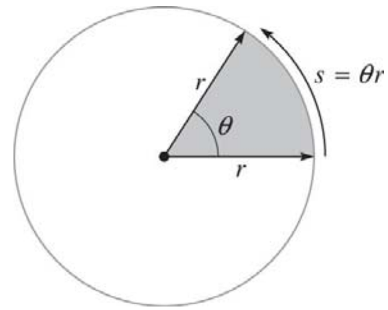
- **Arc Length:**

The arc length s is related to the angle θ (in radians = rad) as follows:

$$s = r\theta$$

$$\theta = \frac{s}{r}$$

(360º = 2π rad)

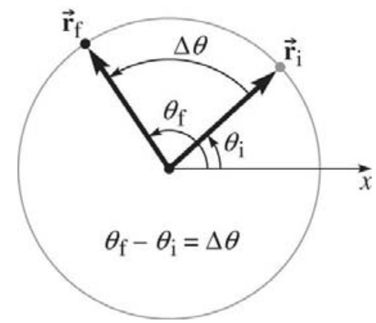


- **Angular Displacement and Angular Velocity:**

$$\Delta\theta = \theta_f - \theta_i$$

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

(radians/second)

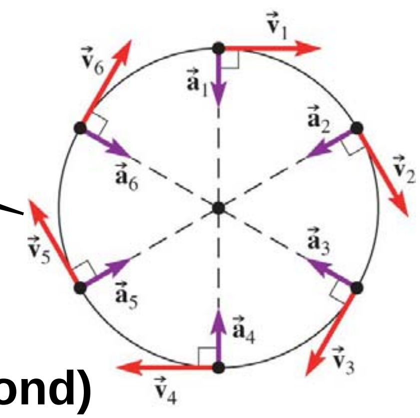


- **Tangential Velocity and Angular Velocity:**

$$v_t = |\vec{v}_t| = \frac{ds}{dt} = r|\omega|$$

$$v_t = r\omega$$

Tangential Velocity



- **Period and Frequency (constant ω):**

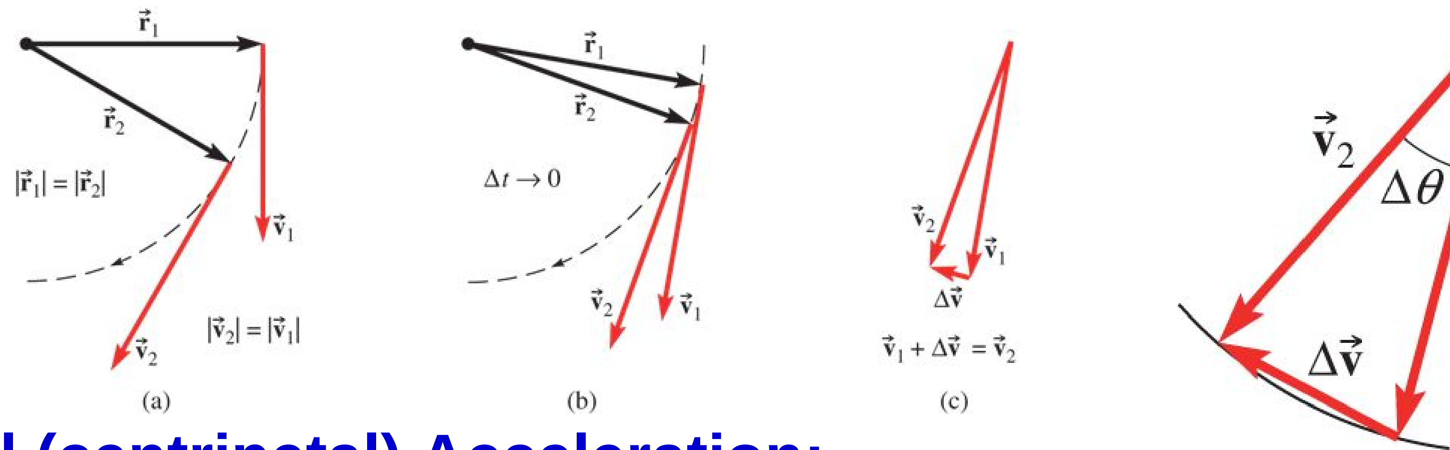
$$C = 2\pi r = v_t T$$

$$T = \frac{2\pi r}{v_t} = \frac{2\pi}{\omega}$$

$$f = \frac{1}{T}$$

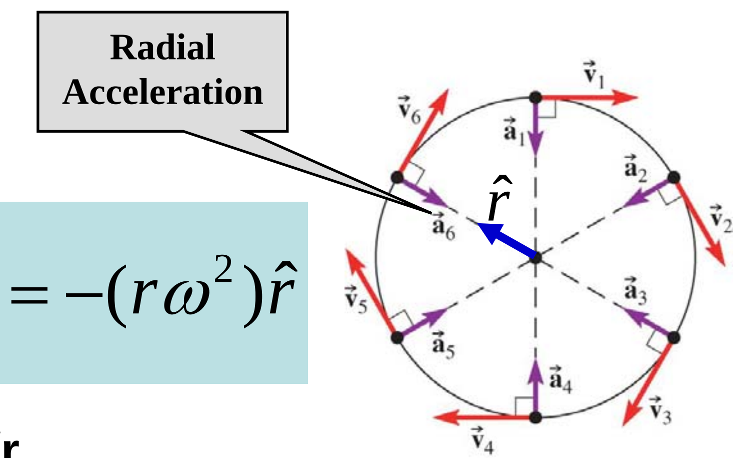
(revolutions/second)

Circular Motion: Radial Acceleration



- **Radial (centripetal) Acceleration:**

Centripetal acceleration = "toward the center"



$$\Delta\vec{v}_t = -(v_t\Delta\theta)\hat{r}$$

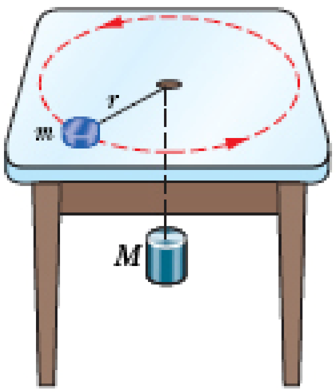
$$\vec{a}_{radial} = \lim_{\Delta t\to 0}\frac{\Delta\vec{v}_t}{\Delta t} = -(v_t\frac{\Delta\theta}{\Delta t})\hat{r} = -(v_t\omega)\hat{r} = -(r\omega^2)\hat{r}$$

$$a_{radial} = |\vec{a}_{radial}| = r\omega^2 = \frac{v_t^2}{r}$$

Magnitude = $(v_t)^2/r$
Direction = toward the center of the circle

Example Problem

- **A puck of mass m slides in a circle of radius r = 0.5 m on a frictionless table while attached to a hanging cylinder of mass M = 2m by a cord through a hole in the table. What speed of the mass m keeps the cylinder at rest?**



Answer: $$v = \sqrt{\frac{Mgr}{m}} \approx 3.13 m/s$$

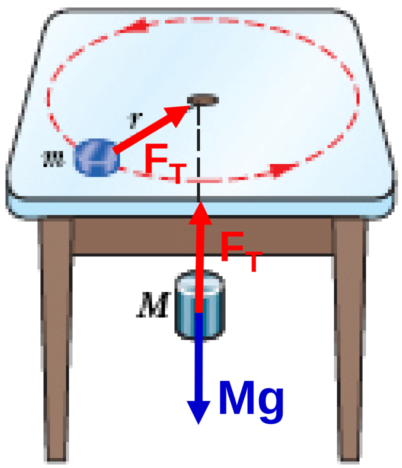
Solution:

$$F_T - Mg = 0$$

$$F_T = ma_{radial} = m\frac{v^2}{r}$$

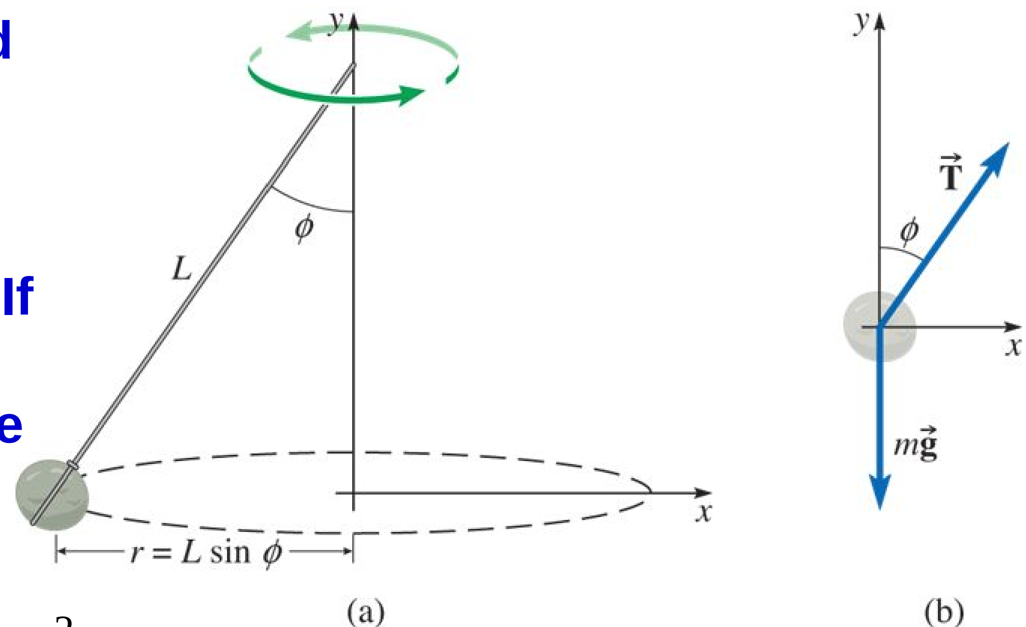
$$m\frac{v^2}{r} = Mg \rightarrow v = \sqrt{\frac{Mgr}{m}} = \sqrt{\frac{2mgr}{m}} = \sqrt{2gr}$$

$$= \sqrt{2(9.8m/s^2)(0.5m)} \approx 3.13 m/s$$



Example: Conical Pendulum

- A stone of mass m is connected to a cord with length L and negligible mass. The stone is undergoing uniform circular motion in the horizontal plane. If the cord makes an angle ϕ with the vertical direction, what is the period of the circular motion?



x-component:

$$T\sin\phi = ma_x = ma_{radial} = mr\omega^2 \quad \textbf{(1)}$$

y-component: $T\cos\phi - mg = ma_y = 0 \longrightarrow T\cos\phi = mg$ **(2)**

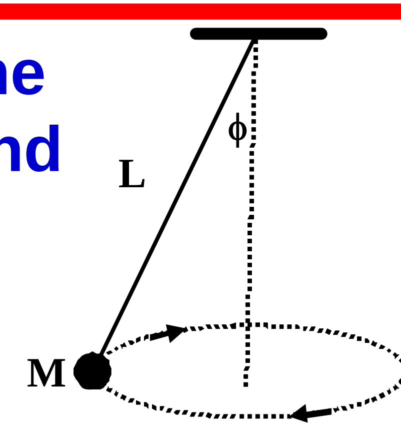
Divide (1) by (2)

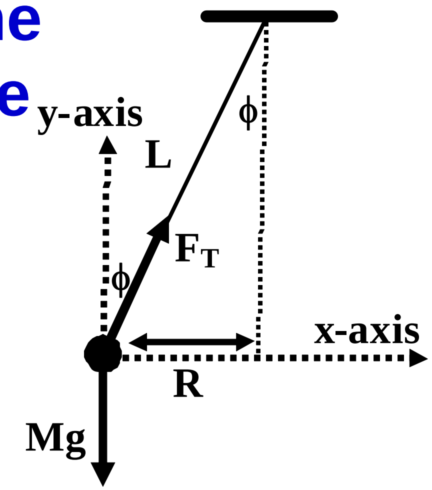
$$\tan\phi = \frac{r\omega^2}{g} = \frac{(L\sin\phi)\omega^2}{g} \rightarrow \omega = \sqrt{\frac{g}{L\cos\phi}}$$

$$T_{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L\cos\phi}{g}}$$

Exam 1 Spring 2012: Problem 20

- **A conical pendulum is constructed from a stone of mass M connected to a cord with length L and negligible mass. The stone is undergoing uniform circular motion in the horizontal plane as shown in the figure. If the cord makes an angle θ = 30° with the vertical direction and the period of the circular motion is 4 s, what is the length L of the cord (in meters)?**





Answer: 4.59

% Right: 34%

$$F_T\cos\phi - Mg = 0 \quad \textbf{(1)}$$

$$F_T\sin\phi = Ma_x = M\frac{v^2}{R} \quad \textbf{(2)}$$

Divide (2) by (1)

$$\tan\phi = \frac{v^2}{Rg}$$

$$T = \frac{2\pi R}{v}$$

$$v^2 = \frac{4\pi^2R^2}{T^2}$$

$$\sin\phi = \frac{R}{L}$$

$$\tan\phi = \frac{4\pi^2R}{gT^2}$$

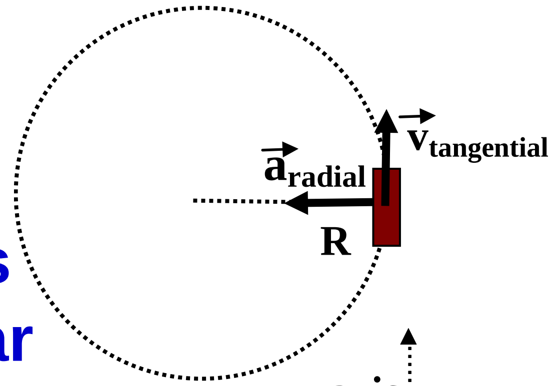
$$R = \frac{gT^2\tan\phi}{4\pi^2}$$

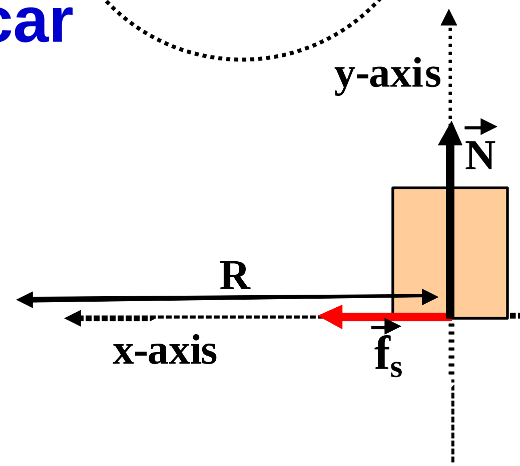
$$L = \frac{R}{\sin\phi}$$

$$L = \frac{gT^2}{4\pi^2\cos\phi} = \frac{(9.8m/s^2)(4s)^2}{4\pi^2\cos(30^\circ)} \approx 4.59m$$

Example Problem: Unbanked Curves

- **A car of mass M is traveling in a circle with radius R on a flat highway with speed v. If the static coefficient of friction between the tires and the road is μ_s, what is the maximum speed of the car such that it will not slide?**





x-component: $f_s = Ma_x = Ma_{radial} = M\dfrac{v^2}{R}$

y-component: $N - Mg = Ma_y = 0$

$$fs \le \mu_s N$$

$$v_{\max}^2 = \frac{R}{M}(f_s)_{\max} = \frac{R}{M}(\mu_s N) = \frac{R}{M}(\mu_s Mg) = \mu_s gR$$

$$v_{\max} = \sqrt{\mu_s gR}$$

Exam 2 Spring 2011: Problem 4

- **Near the surface of the Earth, a car is traveling at a constant speed v around a flat circular race track with a radius of 50 m. If the coefficients of kinetic and static friction between the car's tires and the road are μ_k = 0.1, μ_s = 0.4, respectively, what is the maximum speed the car can travel without slipping?**

Answer: 14 m/s
% Right: 74%

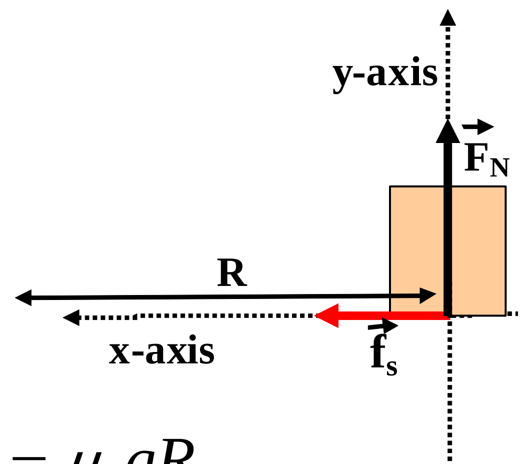
$$f_s = Ma_x = Ma_{radial} = M\frac{v^2}{R}$$

$$F_N - Mg = Ma_y = 0$$

$$fs \le \mu_s F_N$$

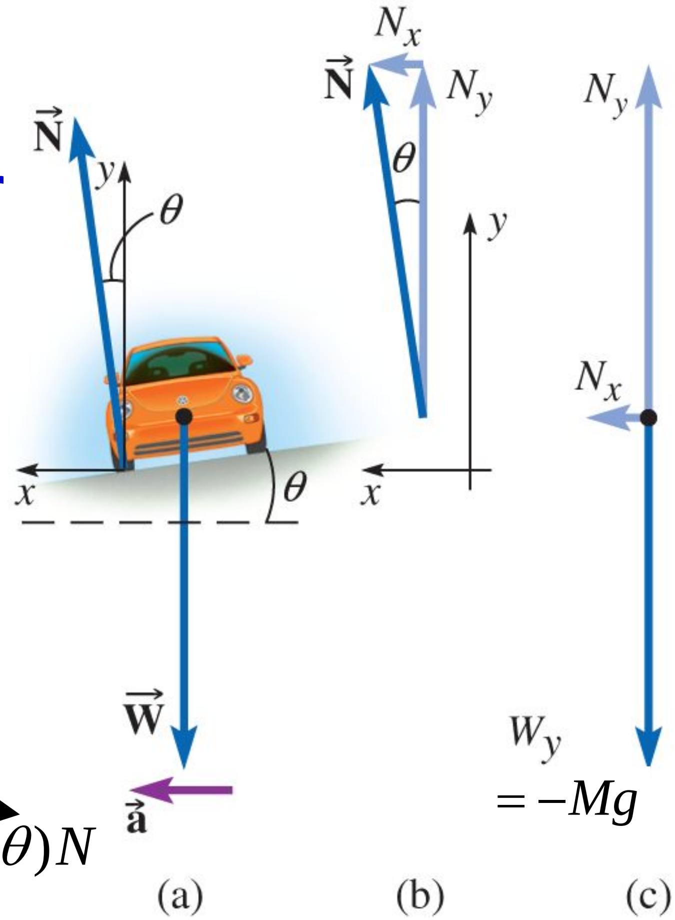
$$v_{\max}^2 = \frac{R}{M}(f_s)_{\max} = \frac{R}{M}(\mu_s F_N) = \frac{R}{M}(\mu_s Mg) = \mu_s gR$$

$$v_{\max} = \sqrt{\mu_s gR} = \sqrt{(0.4)(9.8m/s^2)(50m)} = 14m/s$$



Example Problem: Banked Curves

- **If the car in the previous problem is traveling on a banked road (angle θ), what is the maximum speed of the car such that it will not slide?**



x-component:

$$f_s \cos\theta + N \sin\theta = Ma_{radial} = M\frac{v^2}{R}$$

y-component: $N\cos\theta - f_s \sin\theta - Mg = Ma_y = 0$

$$N = \frac{Mg}{\cos\theta - \mu_s \sin\theta} \qquad (f_s)_{max} = \mu_s N$$

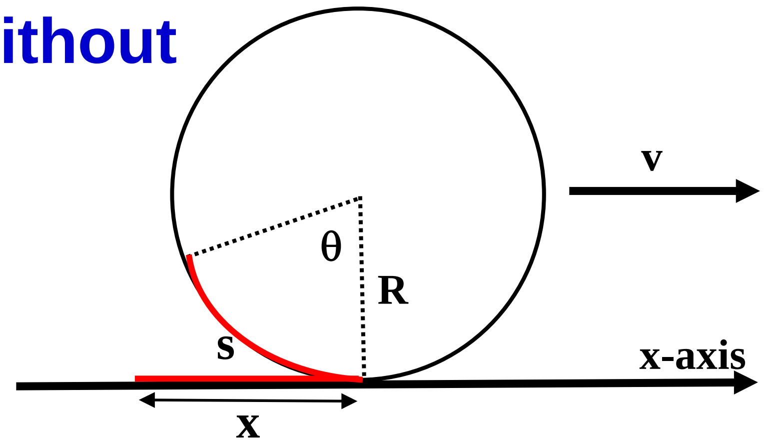
$$v_{max}^2 = \frac{R}{M}\left[(f_s)_{max}\cos\theta + N\sin\theta\right] = \frac{R}{M}(\mu_s \cos\theta + \sin\theta)N$$

$$= Rg\frac{(\mu_s \cos\theta + \sin\theta)}{(\cos\theta - \mu_s \sin\theta)} = Rg\frac{(\mu_s + \tan\theta)}{(1-\mu_s \tan\theta)}$$

$$v_{max} = \sqrt{\frac{Rg(\mu_s + \tan\theta)}{(1-\mu_s \tan\theta)}}$$

Rolling Without Slipping:
Rotation & Translation

- If a cylinder of radius R rolls without slipping along the x-axis then:



$$x = s = r\theta$$

$$v = \frac{dx}{dt} = R\frac{d\theta}{dt} = R\omega$$

Translational Speed

Rotational Speed