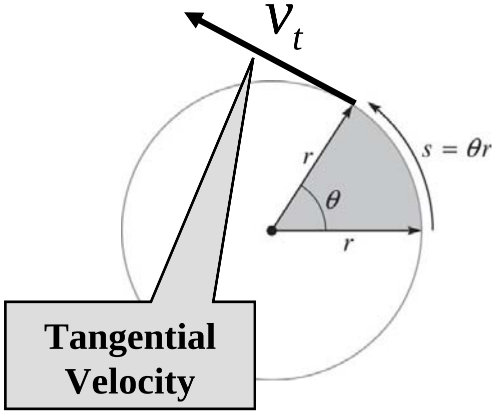
Circular Motion
Tangential & Angular Acceleration



- **Tangential Acceleration:**

The arc length s is related to the angle θ (in radians = rad) as follows:

$$s = r\theta$$

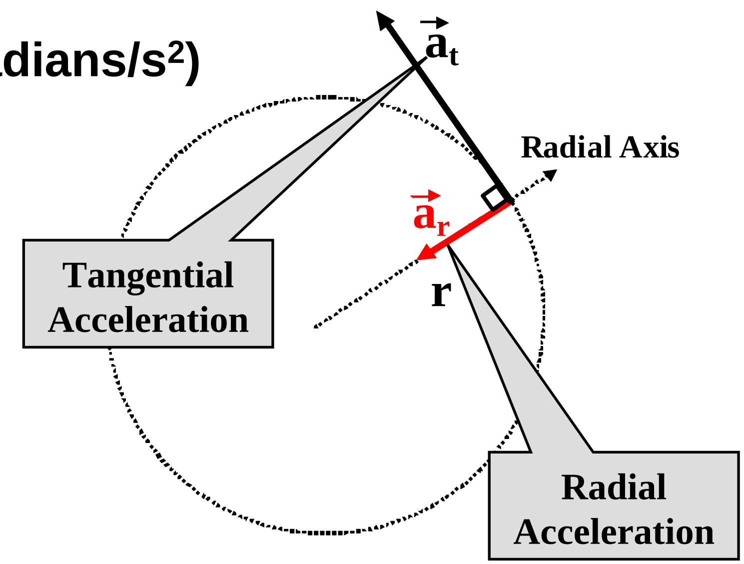
The tangential velocity v_t is related to the angular velocity ω as follows:

$$v_t = r\omega$$

The tangential acceleration a_t is related to the angular acceleration α as follows:

$$a_t = \frac{dv_t}{dt} = r\frac{d\omega}{dt} = r\alpha \qquad \alpha = \lim_{\Delta t \to 0}\frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

(radians/s²)



- **Overall Acceleration:**

$$\vec{a}_{tot} = \vec{a}_{radial} + \vec{a}_t = -a_{radial}\hat{r} + a_t\hat{\theta}$$

$$a_{tot} = \left|\vec{a}_{tot}\right| = \sqrt{a_{radial}^2 + a_t^2}$$

Angular Equations of Motion

- **Angular Equations of Motion (constant α):**

If the angular acceleration α is constant then

$$\alpha(t) = \alpha$$ **(radians/s²)**

$$\omega(t) = \omega_0 + \alpha t$$ **(radians/s)**

$$\theta(t) = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$ **(radians)**

$$\omega^2(t) - \omega_0^2 = 2\alpha\left(\theta(t) - \theta_0\right)$$

$$a_{radial}(t) = r\omega^2(t)$$ **(m/s²)**

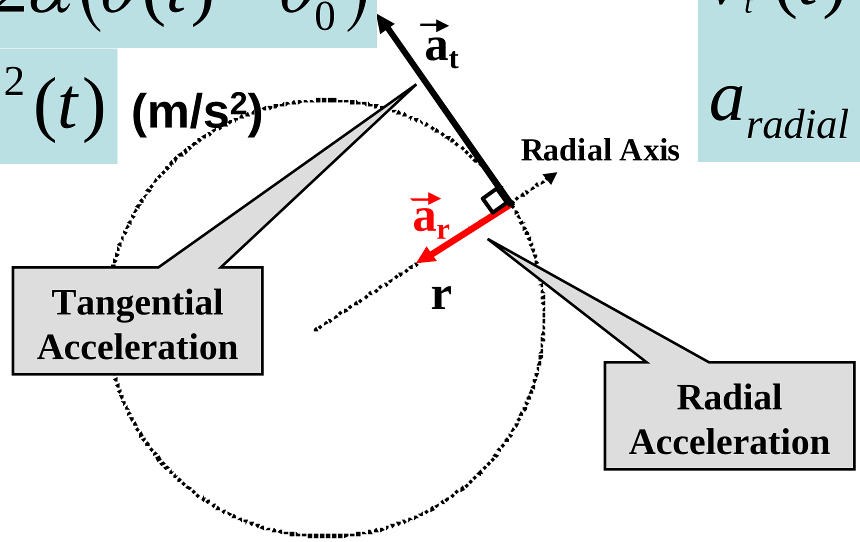
$$a_t(t) = r\alpha$$ **(m/s²)**

$$v_t(t) = v_{t0} + a_t t$$ **(m/s)**

$$s(t) = s_0 + v_{t0}t + \tfrac{1}{2}a_t t^2$$ **(m)**

$$v_t^2(t) - v_{t0}^2 = 2a_t\left(s(t) - s_0\right)$$

$$a_{radial}(t) = v_t^2(t)/r$$ **(m/s²)**



Angular Equations of Motion

- **Angular Equations of Motion (constant α):**

Let N = Number of revolutions (rev)

$$N(t) = \frac{\theta(t)}{2\pi}$$

Let f = Number of revolutions per second

$$f(t) = \frac{\omega(t)}{2\pi}$$ **(frequency)**

$$\alpha(t) = \alpha$$ **(rad/s²)**

$$\omega(t) = \omega_0 + \alpha t$$ **(rad/s)**

$$\theta(t) = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$ **(rad)**

$$\omega^2(t) - \omega_0^2 = 2\alpha\left(\theta(t) - \theta_0\right)$$

$$\frac{\alpha}{2\pi}$$ **(rev/s²)**

$$f(t) = f_0 + (\tfrac{\alpha}{2\pi})t$$ **(rev/s)**

$$N(t) = N_0 + f_0 t + \tfrac{1}{2}(\tfrac{\alpha}{2\pi})t^2$$ **(rev)**

$$f^2(t) - f_0^2 = 2(\tfrac{\alpha}{2\pi})\left(N(t) - N_0\right)$$

Angular Equations: Examples

- **A disk rotates about its central axis starting from rest at t = 0 and accelerates with constant angular acceleration. At one time it is rotating at 4 rev/s; 60 revolutions later, its angular speed is 16 rev/s. Starting at t = 0, what is the time required to complete 64 revolutions?**

$$\frac{\alpha}{2\pi} = \frac{f^2(t) - f_0^2}{2(N(t) - N_0)} = \frac{(16rev/s)^2 - (4rev/s)^2}{2(60rev)} = 2rev/s^2$$

Answer: t = 8 seconds

$$N(t) - N_0 = \tfrac{1}{2}(\tfrac{\alpha}{2\pi})t^2 \qquad t = \sqrt{\frac{2(N(t) - N_0)}{(\frac{\alpha}{2\pi})}} = \sqrt{\frac{2(64rev)}{(2rev/s^2)}} = 8s$$

- **An astronaut is being tested in a centrifuge. The centrifuge has a radius R and, in starting from rest at t = 0, rotates with a constant angular acceleration α = 0.25 rad/s^2. At what time t > 0 is the magnitude of the tangential acceleration equal to the magnitude of the radial acceleration (*i.e.* centripetal acceleration)?**

$$a_{radial}(t) = R\omega^2(t) = R\alpha^2 t^2 = a_t = R\alpha$$

Answer: t = 2 seconds

$$t = \sqrt{\frac{1}{\alpha}} = \sqrt{\frac{1}{0.25rad/s^2}} = 2s$$

Exam 2 Spring 2011: Problem 2

- **A race car accelerates uniformly from a speed of 40 m/s to a speed of 58 m/s in 6 seconds while traveling around a circular track of radius 625 m. When the car reaches a speed of 50 m/s what is the magnitude of its total acceleration (in m/s^2)?**

Answer: 5

% Right: 49%

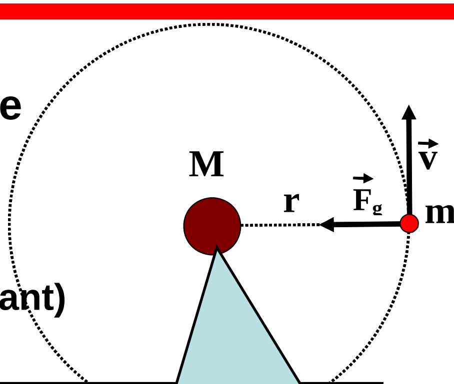
$$a_t = \frac{v_2 - v_1}{t_2 - t_1} = \frac{(58m/s) - (40m/s)}{6s} = 3m/s$$

$$a_r = \frac{v^2}{R} = \frac{(50m/s)^2}{625m} = 4m/s$$

$$a_{tot} = \sqrt{a_t^2 + a_r^2} = 5m/s$$

Gravitation: Circular Orbits (M >> m)

For circular orbits the gravitational force is perpendicular to the velocity and hence the speed of the mass m is constant. The force F_g is equal to the mass times the radial (*i.e.* centripetal) acceleration as follows:



Assume M >> m so that the position of M is fixed!

$$F_g = \frac{GmM}{r^2} = ma_{radial} = m\frac{v^2}{r} = mr\omega^2$$

$$r = \frac{GM}{v^2}$$ **(radius of the orbit, constant)**

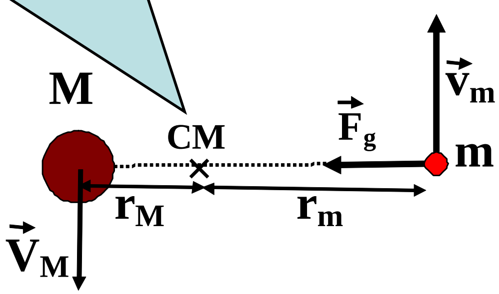
$$v = \sqrt{\frac{GM}{r}}$$ **(speed, constant)**

$$\omega = \sqrt{\frac{GM}{r^3}}$$ **(angular velocity, constant)**

For circular orbits r, v, and ω are also constant.

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{GM}} = 2\pi\sqrt{\frac{r^3}{GM}}$$ **(period of rotation)**

In general both masses rotate about the center-of-mass and the formulas are more complicated!



- **Kepler's Third Law:**

$$T^2 = \frac{4\pi^2 r^3}{GM}$$

The period squared is proportional to the radius cubed.

Circular Orbits: Example

- **Two satellites are in circular orbit around the Earth. The first satellite has mass M_1 and is travelling in a circular orbit of radius R_1. The second satellite has mass $M_2 = M_1$ is travelling in a circular orbit of radius $R_2 = 4R_1$. If the first satellite completes one revolution of the Earth in time T, how long does it take the second satellite to make one revolution of the Earth?**

Answer: 8T

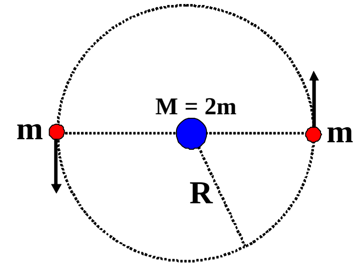
$$T_1^2 = \frac{4\pi^2 R_1^3}{GM_1}$$

$$T_2^2 = \frac{4\pi^2 R_2^3}{GM_2} = \frac{4\pi^2 (4R_1)^3}{GM_1} = 64\frac{4\pi^2 R_1^3}{GM_1} = 64T_1^2$$

$$T_2 = 8T_1 = 8T$$

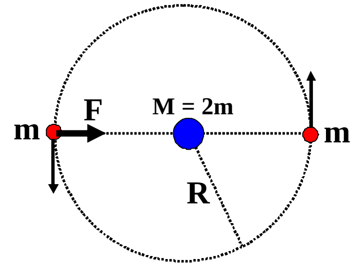
Circular Orbits: Example

- Two diametrically opposed masses m revolve around a circle of radius R. A third mass M = 2m is located at the center of the circle. What is the period T of rotation for this system of three masses?



Answer: $T = \frac{4\pi}{3}\sqrt{\frac{R^3}{Gm}}$

$$F_{grav} = \frac{GmM}{R^2} + \frac{Gm^2}{(2R)^2} = \frac{Gm}{R^2}\left(M + \tfrac{1}{4}m\right) = ma_{radial} = m\frac{v^2}{R}$$



$$v = \sqrt{\frac{G(M + \frac{1}{4}m)}{R}}$$

$$T = \frac{2\pi R}{v} = 2\pi R\sqrt{\frac{R}{G(M + \frac{1}{4}m)}} = 2\pi R\sqrt{\frac{R}{G(2m + \frac{1}{4}m)}} = 2\pi R\sqrt{\frac{4R}{9Gm}} = \frac{4\pi}{3}\sqrt{\frac{R^3}{Gm}}$$