Chapter 7

Linear Momentum

Linear Momentum of a Single Particle $\vec{p}$

- Linear momentum:

$$\vec{p} = m\vec{v}$$

  - It is a measure of the particle's motion
  - It is a vector, similar to the velocity

$$p_x = m v_x \quad p_y = m v_y \quad p_z = m v_z$$

  - It also depends on the mass of the object, similar to the kinetic energy.

Newton's 2nd Law for a Single Body.

- The time rate of the change of the momentum of a particle is equal to the net force acting on the particle.

$$\vec{F}_{net} = m\vec{a} = m\frac{\Delta\vec{v}}{\Delta t} = \frac{\Delta(m\vec{v})}{\Delta t} = \frac{\Delta\vec{p}}{\Delta t}$$

- The momentum formulation of Newton's law is more inclusive as it is applicable to bodies and systems with variable mass.

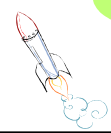
$$\vec{F}_{net}\Delta t = \Delta\vec{p}$$

A few pointers:

- The linear momentum of an object changes (and therefore there is a net force acting on it) if
  - The velocity changes in magnitude

  - The velocity changes in direction

  - The mass of the body changes

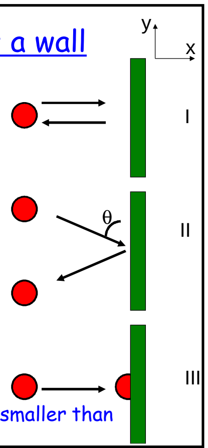


Example- a ball hits a wall

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = -mv\hat{i} - mv\hat{i} = -2mv\hat{i} = -2\vec{p}$$

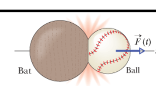
$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i$$
$$\Delta p_x = -mv\sin\theta - mv\sin\theta = -2mv\sin\theta$$
$$\Delta p_y = -mv\cos\theta + mv\cos\theta = 0$$

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = 0 - mv\hat{i} = -mv\hat{i} = -\vec{p}$$



If the ball sticks to the wall the force is smaller than if it bounces.

Collisions



- Before and after the collision the momentum of a body changes.
- There is a net force acting on the body.

$$\vec{F}_{net}\Delta t = \Delta\vec{p}$$

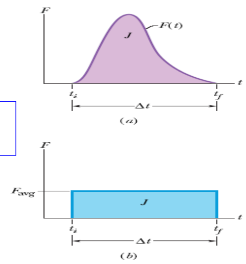
- The collision takes time (although quite small).
- The force varies during the time of the collision.
- The total change of the momentum of the body during the collision is:

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = F_{net}\Delta t$$ ← Impulse

Impulse for a single collision

- The impulse of a force is a vector.

$$\vec{F}\Delta t = \text{area under the curve}$$



The area below the two curves is the same!

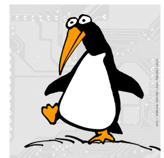
$$\text{same impulse } \vec{F}\Delta t$$

$$\vec{F}\Delta t = \Delta\vec{p} = \vec{p}_f - \vec{p}_i$$

The change of the linear momentum is equal to the impulse of the acting force.

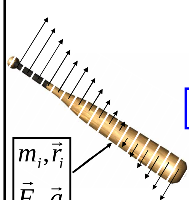
The point mass

- The point mass is a physical approximation of a real object that has dimensions.
- How good is this this approximation???
- Can we apply Newton's laws and all we learned so far to a big, extended object, or to a system of objects that are not even attached to each other?



System of particles

- A body can have a complicated shape
- We can represent the body as a sum of smaller parts
- Apply Newton's law to each part.



$m_i, \vec{r}_i$

$\vec{F}_i, \vec{a}_i$

$$\vec{F}_i = m_i\vec{a}_i$$

$$\vec{F} = \sum_{i=1}^{n}\vec{F}_i = \sum m_i\vec{a}_i$$

$$\vec{F} = M\vec{a}_{CM} \Rightarrow$$

$$\vec{r}_{CM} = \frac{1}{M}\sum m_i\vec{r}_i$$

$$M = \sum m_i$$

The Center of Mass

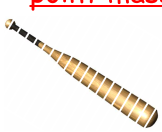
- The center of mass (CM) of a system of particles is the point that moves as though
  - the system's mass is concentrated in this point and
  - all external forces are applied there.

$$\vec{r}_{CM} = \frac{1}{M}\sum m_i\vec{r}_i$$

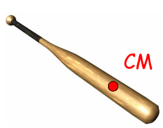
$$M = \sum_{i=1}^{n} m_i$$

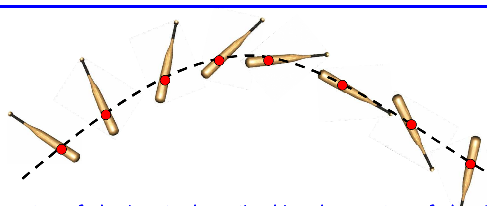
- Important note:
  - We can substitute the entire system with a single point!
  - CM is a geometrical point (it might be outside of the body).

We can replace the object with point mass at the center of mass!



$\Rightarrow$



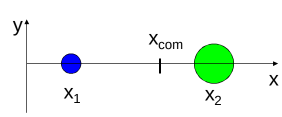


The motion of the bat is described by the motion of the CM.

Example: 2 masses

- Two particles have masses m_1 and m_2 and positions x_1 and x_2 respectively. Find the center of mass of the system.

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$



$$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$1) m_1 = m_2 \Rightarrow x_{CM} = 0.5(x_1 + x_2)$$

$$2) m_2 >> m_1 \Rightarrow x_{CM} \approx x_2$$

Center of Mass of a System

System of particles

$$\vec{r}_{CM} = \frac{1}{M}\sum m_i \vec{r}_i$$

$$M = \sum m_i$$

$$\Rightarrow$$

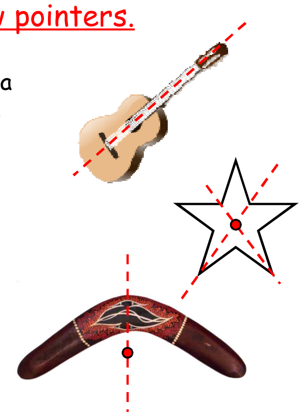
$$x_{CM} = \frac{1}{M}\sum m_i x_i$$

$$y_{CM} = \frac{1}{M}\sum m_i y_i$$

$$z_{CM} = \frac{1}{M}\sum m_i z_i$$

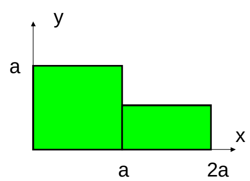
A few pointers.

- If a body has a symmetry and it has a uniform density then the CM is on the line of symmetry.
- The center of symmetry coincides with the CM.
- The CM might be outside the object



Example

- Find the CM of a 4x8 uniform sheet of plywood with the upper right quadrant removed (a=4ft).



$$m_1 = 2M; (x_1, y_2) = (\frac{a}{2}, \frac{a}{2})$$

$$m_2 = M; (x_2, y_2) = (\frac{3a}{2}, \frac{a}{4})$$

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{2M\cdot(a/2) + M\cdot(3a/2)}{2M + M} = \frac{5a}{6} = \frac{10}{3}\,ft$$

$$y_{CM} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = \frac{2M\cdot(a/2) + M\cdot(a/4)}{2M + M} = \frac{5a}{12} = \frac{5}{3}\,ft$$

Newton's Second Law for a System of Particles

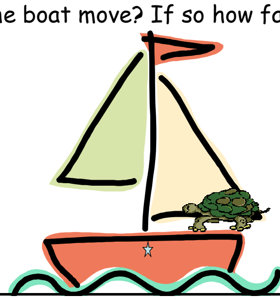
- The sum of all external forces F_{net} acting on the system is equal to the product of the total mass M of the system and the acceleration of the center of mass a_{CM}.

$$\vec{F}_{net} = M\vec{a}_{CM}$$

- F_{net} - sum of forces that are external to the system (?)
- M is the total mass
- It does not give information on the acceleration of the individual objects of the system.
- If no external forces are present the center of mass will stay at rest or move with constant velocity!

Example Problem

- A 10 m long boat stays at rest in a quite lake. A turtle with a mass m, is originally standing at the right end of the boat. The turtle walks to the other end of the boat. Does the boat move? If so how far?



Solution

- Yes the boat will move, but the center of mass of the boat+turtle system will stay at rest.

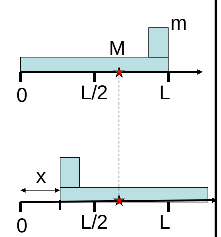
before $x_{CM} = \dfrac{0.5ML + mL}{M + m}$

after $x_{CM} = \dfrac{M(0.5L + x) + mx}{M + m}$

$$M(0.5L + x) + mx = 0.5ML + mL$$

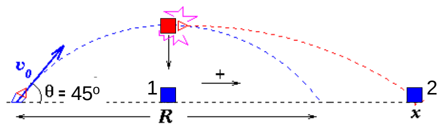
$$Mx + mx = mL$$

$$x = \frac{m}{M + m}L$$



Example Problem

- A cannon shell is fired with an initial velocity of 50 m/s at an angle 45° to the horizontal. When the shell is at the highest point of its trajectory, it explodes into two pieces, m_1 and m_2. m_1 falls straight down from the highest point after the explosion. If $m_1 = 2m_2$, where does m_2 land relative to the initial firing point?



Solution

- The CM will follow the trajectory of the original cannon shell and final x_{cm} can be found with the range equation.

$$x_{CM} = R = \frac{v_0^2}{g}\sin 2\theta = 255m \qquad m_1 = 2m_2$$

$$x_{CM} = \frac{0.5m_1 R + m_2 x}{m_1 + m_2} = \frac{m_2 R + m_2 x}{2m_2 + m_2} = \frac{R + x}{3}$$

$$\frac{R + x}{3} = R$$

$$x = 3R - R = 2R = 510m$$

Linear Momentum of a System of particles

- The vector sum of the linear momentum of all particles in the system!

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + ... + \vec{p}_n$$

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + ... + m_n\vec{v}_n = \sum m_i\vec{v}_i$$

Can show with some work

$$\vec{P} = M\vec{v}_{CM}$$

CM velocity

Total mass

Newton's 2nd Law for a system of particles

- The net force (the vector sum of all external forces) acting on the system of particles is equal to the rate of change of the the total linear momentum of the system.

$$\vec{F}_{net}\Delta t = \Delta\vec{P}$$

- The linear momentum of a system remains constant if no net external force is acting on the system.

Conservation of Linear Momentum

- If a system is closed and isolated the momentum of the system is constant.

$$\vec{P} = \text{constant} \qquad \Delta\vec{P} = 0$$

- If no net external force is acting on the system then the momentum is conserved

$$\vec{p}_{1f} + \vec{p}_{2f} + ... + \vec{p}_{nf} = \vec{p}_{1i} + \vec{p}_{2i} + ... + \vec{p}_{ni}$$

- If the net force on a closed system is zero in a given direction (axis) the momentum in this direction cannot change.

Example

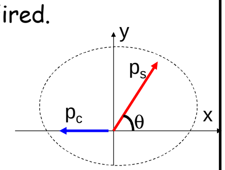
- A cannon fires a shell at 60° angle to the horizontal. The shell initial velocity is v_s=100 m/s. If the ratio of the masses of the cannon and the shell is 100 find the velocity of the cannon after the shot was fired.

$$P_{xi} = P_{xf} \quad \Rightarrow \quad 0 = p_c + p_{sx}$$

$$p_c = -p_{sx} = -m_s v_{sx} = -m_s v_s \cos\theta$$

$$p_c = m_c v_c \qquad v_c = -\frac{m_s}{m_c} v_s \cos\theta$$

$$v_c = -\cos 60 = -0.5\ m/s$$



External forces in y: F_g F_N

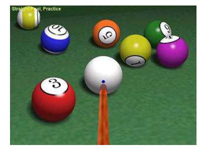
Momentum is conserved in x

Momentum and Kinetic Energy conservation Laws

- Mechanical energy conservation
  - Conditions:
    - isolated system, (no external forces)
    - closed system, (do not lose bodies)
    - conservative forces. (no friction, drag…)
- Linear momentum conservation
  - Conditions:
    - isolated system,
    - closed system.

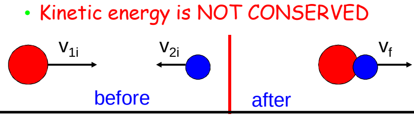
Elastic Collisions

- Elastic collisions: no deformations, no heating of the bodies, no explosions ,no crashing sound.
  - Examples: two balls collide and separate after the collision (pool games).
  - Conservation laws:
    - linear momentum is conserved
    - kinetic energy is conserved.



Inelastic collisions

- Inelastic collisions: there is deformation, or heating, or explosion, or sound …
  - Examples: a bullet hits a target and goes through the target;
  - Completely inelastic collisions: two balls collide and stick together; a bullet hits a target and remains inside; a ball is thrown in a cart and stays there…
  - Conservation laws:
    - Linear momentum is conserved
    - Kinetic energy is NOT CONSERVED



Inelastic Collisions in 1D

- What happens before and after the collision?
  - Linear momentum of the system is conserved
  - Kinetic energy of the system is not conserved

$$p_{1i} + p_{2i} = p_{1f} + p_{2f}$$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

- We have only 1 equation.
- We have 6 parameters. We need to know 5 of them to find the 6th.
- If we know all initial conditions (masses of the bodies, velocities before the collision) we still cannot find both velocities after the collision!

Completely inelastic collision 1D

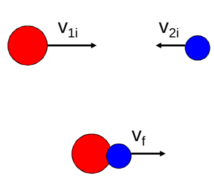
- What happens before and after the collision?
  - Linear momentum of the system is conserved
  - Kinetic energy of the system is not conserved
  - After the collision the two bodies move together with a common velocity V.

$$p_{1i} + p_{2i} = p_{1f} + p_{2f}$$

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2)V$$

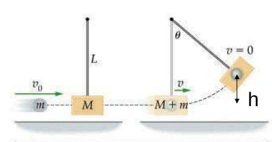
$$V = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2}$$

$$V_{CM} = V$$



Example Problem: ballistic pendulum

- If you know the height and the masses, find the initial velocity of the bullet.



During the collision: Momentum conservation

$$m v_0 = (m + M)V$$

$$v_0 = \frac{m + M}{m} V$$

After the collision: Momentum conservation

$$\frac{1}{2}(m + M)V^2 = (m + M)gh$$

$$V = \sqrt{2gh}$$

$$v_0 = \frac{m + M}{m}\sqrt{2gh}$$

Elastic Collisions in 1D

- What happens before and after the collision?
  - Linear momentum of the system is conserved
  - Kinetic energy of the system is conserved

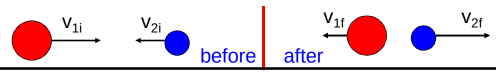
$$p_{1i} + p_{2i} = p_{1f} + p_{2f}$$

$$m_1 \upsilon_{1i} + m_2 \upsilon_{2i} = m_1 \upsilon_{1f} + m_2 \upsilon_{2f}$$

$$K_{1i} + K_{2i} = K_{1f} + K_{2f}$$

$$\frac{1}{2} m_1 \upsilon_{1i}^2 + \frac{1}{2} m_2 \upsilon_{2i}^2 = \frac{1}{2} m_1 \upsilon_{1f}^2 + \frac{1}{2} m_2 \upsilon_{2f}^2$$

- If we know the initial velocities and the masses then we can find both final velocities.



Example: stationary target playing pool

- The target does not move $v_{2i}=0$

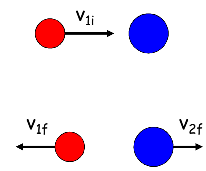
$$m_1 \upsilon_{1i} = m_1 \upsilon_{1f} + m_2 \upsilon_{2f}$$

$$m_1 \upsilon_{1i}^2 = m_1 \upsilon_{1f}^2 + m_2 \upsilon_{2f}^2$$

$$\Downarrow$$

$$\upsilon_{1f} = \frac{m_1 - m_2}{m_1 + m_2} \upsilon_{1i}$$

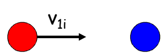
$$\upsilon_{2f} = \frac{2m_1}{m_1 + m_2} \upsilon_{1i}$$



Special cases

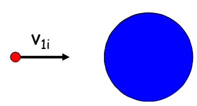
- $m_1=m_2$ the two masses are the same

$$\upsilon_{1f} = 0 \qquad \upsilon_{2f} = \upsilon_{1i}$$



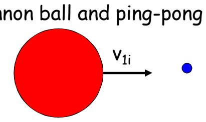
- $m_1 \ll m_2$ massive target (like wall)

$$\upsilon_{1f} \approx -\upsilon_{1i} \qquad \upsilon_{2f} \approx \frac{2m_1}{m_2} \upsilon_{1i}$$



- $m_1 \gg m_2$ massive projectile (cannon ball and ping-pong ball)

$$\upsilon_{1f} \approx \upsilon_{1i} \qquad \upsilon_{2f} \approx 2\upsilon_{1i}$$



Example: Moving target

- Both bodies are moving before the collision.
- This is the most general case and you can derive the rest of the cases from these formulas

$$\upsilon_{1f} = \frac{m_1 - m_2}{m_1 + m_2} \upsilon_{1i} + \frac{2m_1}{m_1 + m_2} \upsilon_{2i}$$

$$\upsilon_{2f} = \frac{2m_1}{m_1 + m_2} \upsilon_{1i} + \frac{m_1 - m_2}{m_1 + m_2} \upsilon_{2i}$$

Elastic Collisions in 2D

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

$$K_{1i} + K_{2i} = K_{1f} + K_{2f}$$

- 2 momentum equations (for x and y)
- 1 energy equation.
- We have total of 8 parameters.
- We need to know 5 of the parameters to determine the rest of them.