Simple Harmonic Motion: SHM

- **Linear Restoring Force:** **Ideal Spring**

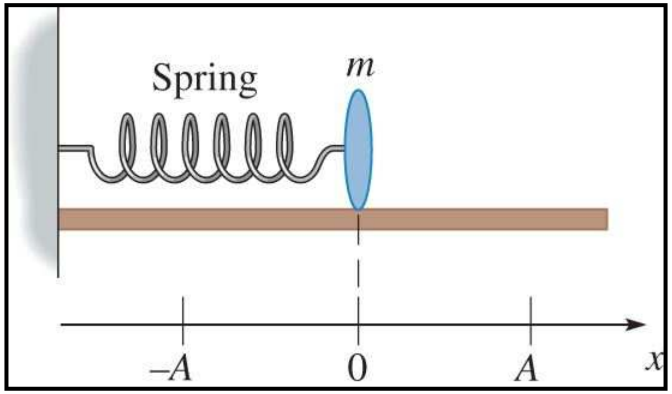
$$F_x = ma_x = -kx$$

Spring Constant k

$$a_x(t) = -\frac{k}{m}x(t)$$

$$a_{max} = \frac{k}{m}x_{max} = \frac{k}{m}A$$

Amplitude A



- **Energy Conservation:**

$$E = KE + U = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2$$ constant (independent of time)

At x_{max} = A v_x = 0 hence

$$E = \tfrac{1}{2}kA^2 = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2$$

$$\frac{m}{k}v_x^2(t) + x^2(t) = A^2$$ (true at any time t)

The maximum speed v_{max} occurs when x = 0.

$$v_{max} = \sqrt{\frac{k}{m}}A$$

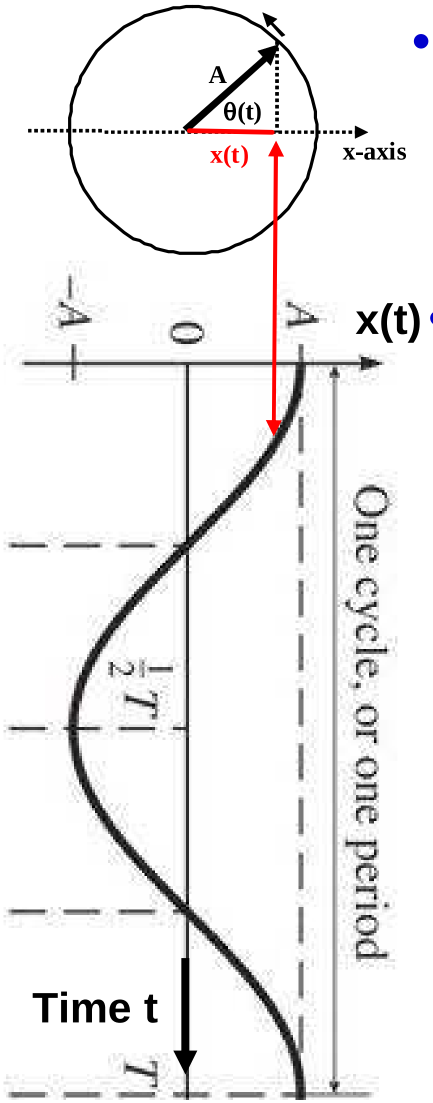
General Solution!

$$x(t) = A\cos(\omega t + \phi)$$
$$v_x(t) = -\omega A\sin(\omega t + \phi)$$
$$a_x(t) = -\omega^2 A\cos(\omega t + \phi)$$

$$\omega_{spring} = \sqrt{\frac{k}{m}}$$

The phase angle ϕ determines where the mass m is at t = 0, x(t=0) = A$\cos\phi$. If x(t=0) = A then ϕ = 0.

Uniform Circular Motion & SHM



- **Uniform Circular Motion:** $\theta(t) = \omega t$

Project uniform circular motion (constant angular velocity ω) of a vector with length A onto the x-axis and you get SHM!

$$x(t) = A\cos[\theta(t)] = A\cos(\omega t)$$

- **Simple Harmonic Motion (SHM):**

If x(t=0) = A then

$$x(t) = A\cos(\omega t) \qquad x_{\max} = A \text{ (Amplitude)}$$

$$v_x(t) = -\omega A\sin(\omega t) \qquad v_{\max} = \omega A$$

$$a_x(t) = -\omega^2 A\cos(\omega t) \qquad a_{\max} = \omega^2 A$$

$$a_x(t) = -\omega^2 x(t) \underset{spring}{=} -\frac{k}{m}x(t)$$

$$\omega \underset{spring}{=} \sqrt{\frac{k}{m}}$$

The period T is the time is takes for one circular revolution:

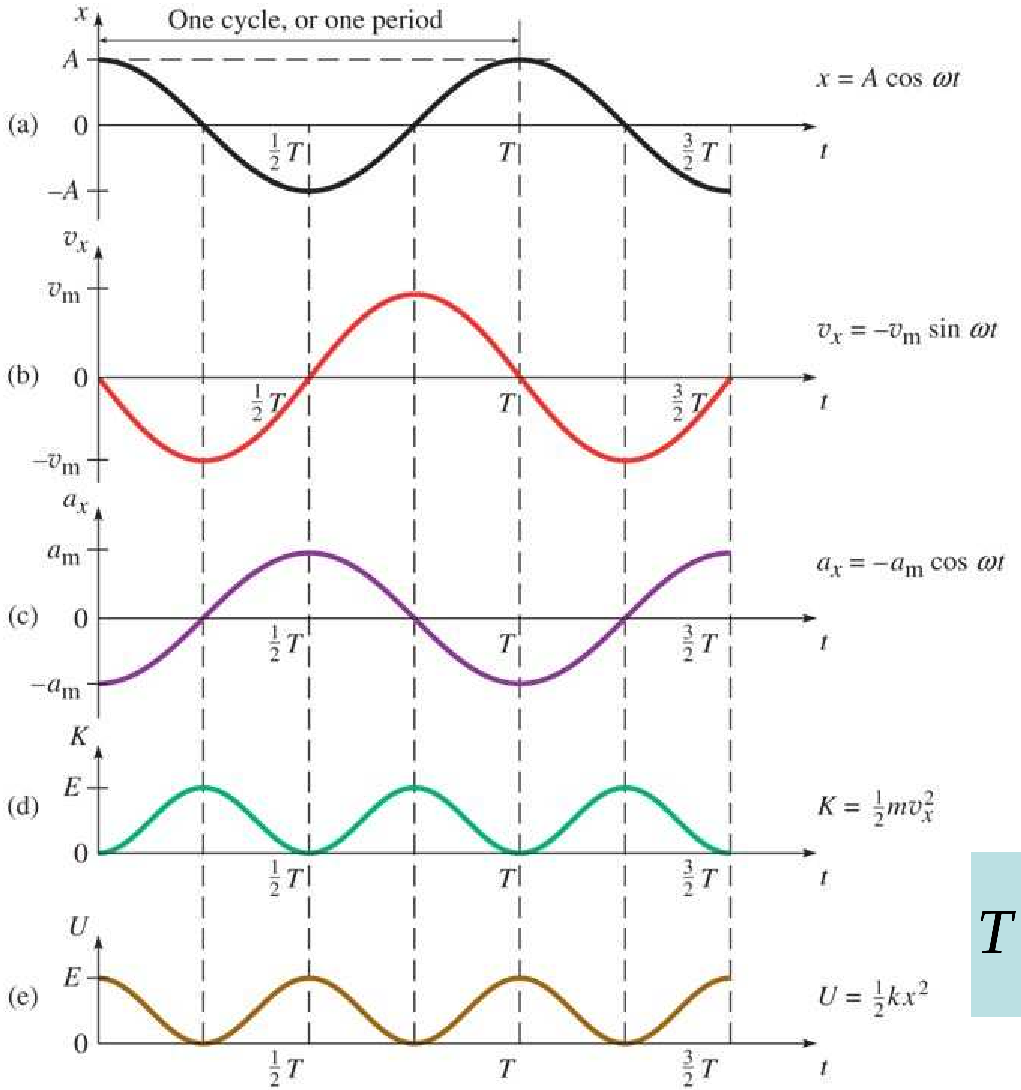
$$T = \frac{2\pi}{\omega} \qquad f = \frac{1}{T} \qquad \omega = 2\pi f$$

T = period (in s)

f = frequency (in Hz)

ω = angular frequency (in rad/sec)

SHM: Graphical Representation



If x(t=0) = A then

$$x(t) = A\cos(\omega t)$$
$$v_x(t) = -v_{max}\sin(\omega t)$$
$$a_x(t) = -a_{max}\cos(\omega t)$$

$$x_{max} = A$$
$$v_{max} = \omega A$$
$$a_{max} = \omega^2 A$$

$$\omega_{spring} = \sqrt{\frac{k}{m}}$$

$$T = \frac{2\pi}{\omega} \qquad f = \frac{1}{T} \qquad \omega = 2\pi f$$

SHM: General Solution

If the acceleration $a_x(t)$ and the position x(t) are related as follows:

$$a_x(t) = -Cx(t)$$ **where C is some constant then**

$$C_{spring} = \frac{k}{m}$$

$$x(t) = A\cos(\omega t + \phi)$$
$$v_x(t) = -v_{max}\sin(\omega t + \phi)$$
$$a_x(t) = -a_{max}\cos(\omega t + \phi)$$

$$x_{max} = A$$
$$v_{max} = \sqrt{C}A$$
$$a_{max} = CA$$

$$\omega = \sqrt{C}$$

$$T = \frac{2\pi}{\sqrt{C}}$$ $$f = \frac{1}{T}$$ $$\omega = 2\pi f$$

If x(t=0) = A then ϕ = 0:

$$x(t) = A\cos(\omega t)$$
$$v_x(t) = -v_{max}\sin(\omega t)$$
$$a_x(t) = -a_{max}\cos(\omega t)$$

If x(t=0) = 0 and v_x(t=0) > 0 then $\phi = \pi/2$:

$$x(t) = A\sin(\omega t)$$
$$v_x(t) = v_{max}\cos(\omega t)$$
$$a_x(t) = -a_{max}\sin(\omega t)$$

$$\cos(A+B) = \cos A\cos B + \sin A\sin B$$

SHM: General Solution

- **Angular Oscillations SHM:**

If the angular acceleration α(t) and the angular position θ(t) are related as follows:

$$\alpha(t) = -C\theta(t)$$

where C is some constant then

$$\theta(t) = A\cos(\xi t + \phi)$$
$$\omega(t) = -\omega_{max}\sin(\xi t + \phi)$$
$$\alpha(t) = -\alpha_{max}\cos(\xi t + \phi)$$

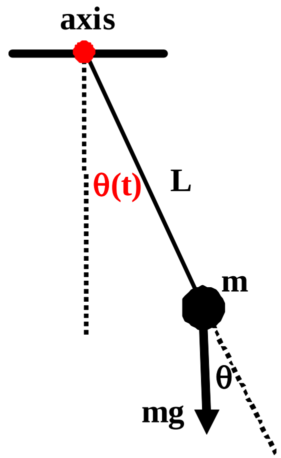
$$\theta_{max} = A$$
$$\omega_{max} = \sqrt{C}A$$
$$\alpha_{max} = CA$$

$$\xi = \sqrt{C}$$

$$T = \frac{2\pi}{\sqrt{C}} \qquad f = \frac{1}{T} \qquad \omega = 2\pi f$$

The Pendulum: Small Oscillations SHM

- **Simple Pendulum:**



Small pendulum bob with mass m on string of lengh L and negligible mass. Calculate the torque about the axis of rotation as follows:

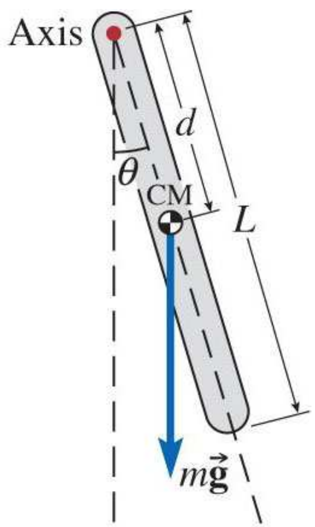
$$\tau = I\alpha = -mgL\sin\theta \qquad \alpha(t) = -\frac{mgL}{mL^2}\sin\theta \underset{\theta<<1}{\rightarrow} -\frac{g}{L}\theta(t)$$

$$I = mL^2$$

$$\alpha(t) = -C\theta(t) \qquad C = \frac{g}{L}$$

SHM with period T given by $T = \frac{2\pi}{\sqrt{C}} = 2\pi\sqrt{\frac{L}{g}}$ **(simple pendulum)**

- **Physical Pendulum:**



Moment of inertia, I, Length L, mass m, distance from axis of rotation to the center-of-mass, dcm. Calculate the torque about the axis of rotation as follows:

$$\tau = I\alpha = -d_{cm}mg\sin\theta \qquad \alpha(t) = -\frac{mgd_{cm}}{I}\sin\theta \underset{\theta<<1}{\rightarrow} -\frac{mgd_{cm}}{I}\theta(t)$$

$$\alpha(t) = -C\theta(t) \qquad C = \omega^2 = \frac{mgd_{cm}}{I}$$

SHM with period T given by $T = \frac{2\pi}{\sqrt{C}} = 2\pi\sqrt{\frac{I}{mgd_{cm}}}$ **(physical pendulum)**

SHM: Example Problems

- **A simple harmonic oscillator consists of a block of mass 2 kg attached to a spring of spring constant 200 N/m. If the speed of the block is 40 m/s when the displacement from equilibrium is 3 m, what is the amplitude of the oscillations?** **Answer: 5m**

$$E = \tfrac{1}{2}kA^2 = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 \qquad A^2 = \frac{m}{k}v_x^2(t) + x^2(t)$$

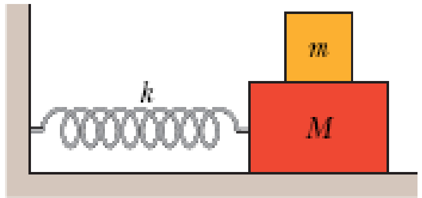
$$A = \sqrt{\frac{m}{k}v_x^2(t) + x^2(t)} = \sqrt{\frac{(2kg)}{(200N/m)}(40m/s)^2 + (3m)^2} = 5m$$

- **A simple pendulum has a length L. If its period is T when it is on the surface of the Earth (gravitational acceleration g), what is its period when it is on the surface of a planet with gravitational acceleration equal to g/4?** **Answer: 2T**

$$T = 2\pi\sqrt{\frac{L}{g}} \qquad T_{new} = 2\pi\sqrt{\frac{L}{(g/4)}} = 2\times 2\pi\sqrt{\frac{L}{g}} = 2T$$

Exam 2 Fall 2011: Problem 52

- In the figure, two blocks (m = 5 kg and M = 15 kg) and a spring (k = 196 N/m) are arranged on a horizontal frictionless surface. If the smaller block begins to slip when the amplitude of the simple harmonic motion is greater than 0.5 m, what is the coefficient of static friction between the two blocks? (Assume that the system is near the surface of the Earth.)



Answer: 0.5
% Right: 26%

$$\omega_{spring} = \sqrt{\frac{k}{(M+m)}}$$

$$f_s = ma_x \le \mu_s F_N \rightarrow a_{max} = \mu_s g$$

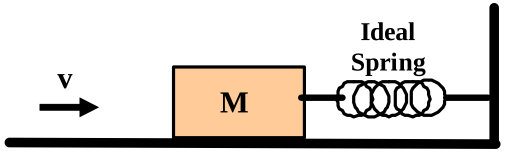
$$F_N = mg$$

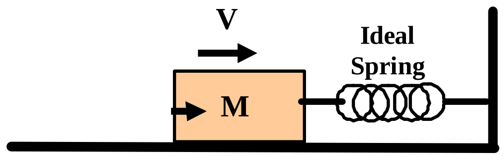
$$a_{max} = \omega_{spring}^2 A = \frac{k}{(M+m)} A$$

$$\mu_s = \frac{k}{(M+m)g} A = \frac{196 N/m}{(20kg)(9.8m/s)}(0.5m) = 0.5$$

Exam 3 Spring 2013: Problem 37

- A block of mass M = 4 kg is at rest on a horizontal frictionless surface and is connected to an ideal spring as shown in the figure. A 2-gram bullet traveling horizontally at 290 m/s strikes the block and becomes embedded in the block. If the period of the subsequent simple harmonic motion of the bullet-block system is 1.73 s, what is the amplitude of the oscillations (in cm)?





Answer: 4.0

% Right: 61%

$$mv = (m+M)V$$

$$V = mv/(m+M)$$

$$\tfrac{1}{2}kA^2 = \tfrac{1}{2}(m+M)V^2 + \tfrac{1}{2}kx^2 \xrightarrow[x=0]{} \tfrac{1}{2}m^2v^2/(m+M)$$

$$T = \frac{2\pi}{\omega}$$

$$\omega = \sqrt{\frac{k}{(m+M)}} \qquad k = (m+M)\omega^2 = 4\pi^2(m+M)/T^2$$

$$A = \sqrt{\frac{m^2v^2}{k(m+M)}} = \sqrt{\frac{m^2v^2T^2}{4\pi^2(m+M)^2}} = \frac{mvT}{2\pi(m+M)} = \frac{(0.002kg)(290m/s)(1.73s)}{2\pi(4.002kg)} \approx 4.0cm$$