Standing waves

$$y_1 = A \sin(kx - \omega t)$$

$$y_2 = A \sin(kx + \omega t)$$

Superposition

$$y = A[\sin(kx - \omega t) + \sin(kx + \omega t)]$$

use same identity as before:

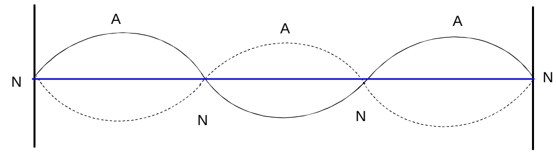
$$y = 2A\left\{\sin\left(\frac{1}{2}[kx - \omega t + kx + \omega t]\right) \cdot \cos\left(\frac{1}{2}[kx - \omega t - kx - \omega t]\right)\right\}$$

$$\boxed{y = 2A \sin(kx) \cos(\omega t)}$$

Not a traveling wave [which would be $\sin(kx - \omega t)$] but a standing wave.

Standing waves

The previous expression is the mathematical form of a standing wave.



A **node** (N) is a point of zero oscillation. An **antinode** (A) is a point of maximum displacement. All points between nodes oscillate up and down.

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Nodes and Antinodes

The nodes occur where y(x,t) = 0.

$$y(x,t) = 2A\cos\omega t \sin kx = 0$$

The nodes are found from the locations where sin kx = 0, which happens when kx = 0, π, 2π,….

$$kx = n\pi \text{ and } n = 0,1,2,\ldots$$

The antinodes occur when sin kx = ± 1; that is where

$$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$kx = \frac{(2n+1)\pi}{2} \text{ and } n = 0,1,2,\ldots$$

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Standing waves on a string, both ends fixed

If the string has a length L, and both ends are fixed, then y(x = 0, t) = 0 and y(x = L, t) = 0 (node at both ends!!)

$$y(x=0,t) \propto \sin k(0) = 0$$

$$y(x=L,t) \propto \sin kL = 0$$

$$kL = n\pi$$

$$\frac{2\pi}{\lambda}L = n\pi$$

The wavelength of a standing wave:

$$\lambda = \frac{2L}{n}$$

where n = 1, 2, 3,…

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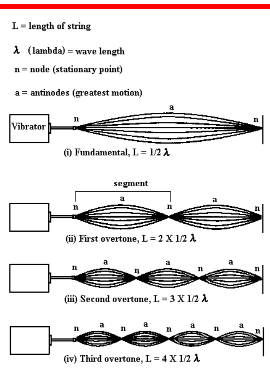
Standing waves cont'd



$$L = \frac{1}{2}\lambda$$

$$L = \lambda$$

$$L = \frac{3}{2}\lambda$$

$$L = 2\lambda$$

Standing waves cont'd

$$\lambda_n = \frac{2L}{n}$$

These are the permitted wavelengths of standing waves on a string; no others are allowed.

The speed of the wave is: $v = \lambda_n f_n$

The allowed frequencies are then: $f_n = \frac{v}{\lambda_n} = \frac{nv}{2L}$ n =1, 2, 3,…

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Natural frequency and Resonance

The n = 1 frequency is called the fundamental frequency (n=2 first overtone, etc.).

$$f_n = \frac{v}{\lambda_n} = \frac{nv}{2L} = n\left(\frac{v}{2L}\right) = nf_1$$

All allowed frequencies (called harmonics) are integer multiples of f_1.

n=1 frequency is also called 1st harmonic. n=2 is called 2nd harmonic, etc.

Example

Example (text problem 11.51): A Guitar's E-string has a length 65 cm and is stretched to a tension of 82 N. It vibrates with a fundamental frequency of 329.63 Hz. Determine the mass per unit length of the string.

For a wave on a string: $v = \sqrt{\frac{F}{\mu}}$

Solving for the linear mass density:

$$\mu = \frac{F}{v^2} = \frac{F}{(\lambda_1 f_1)^2} = \frac{F}{f_1^2 (2L)^2}$$
$$= \frac{(82\ \text{N})}{(329.63\ \text{Hz})^2 (2*0.65\ \text{m})^2} = 4.5\times10^{-4}\ \text{kg/m}$$

Example Problem: Standing Waves

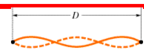
- A string in a grand piano is 2 m long and has a mass density of 1 g/m. If the fundamental frequency of oscillations of the string is 440 Hz, what is the tension in the string (in N)?

Answer: 3097.6

$$v = \sqrt{\frac{F_T}{\mu}} \rightarrow F_T = \mu v^2 \rightarrow F_T = 4\mu L^2 f_1^2 = 4(0.001kg/m)(2m)^2(440/s)^2 = 3097.6N$$

$$f_1 = \frac{v}{2L} \rightarrow v = 2Lf_1$$

Example Problem: Standing Waves



- A nylon guitar string has a linear density of 5 g/m and is under a tension of 200 N. The fixed supports are D = 60 cm apart. The string is oscillating in the standing wave pattern shown in the figure. What is the frequency of the traveling waves whose superposition gives this standing wave?

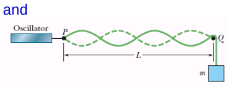
Answer: 500 Hz

$$D = \frac{3\lambda}{2} \rightarrow \lambda = \frac{2D}{3} \qquad v = \sqrt{\frac{F_T}{\mu}}$$

$$f = \frac{v}{\lambda} = \frac{1}{\lambda}\sqrt{\frac{F_T}{\mu}} = \frac{3}{2D}\sqrt{\frac{F_T}{\mu}} = \frac{3}{2(0.6m)}\sqrt{\frac{200N}{(0.005kg/m)}} = 500Hz$$

Example Problem: Standing Waves

- A string, which is tied to a sinusoidal oscillator at P and which runs over a support Q, is stretched by a bloc mass m. The distance L = 2.0 m, the linear mass density of the string μ = 4.9 g/m, and the oscillator frequency f = 100 Hz. The motion at P is in the vertical direction, and its amplitude is small enough for that point to be considered a node. A node also exists at Q. What mass allows the oscillator to set up the second harmonic on the string?



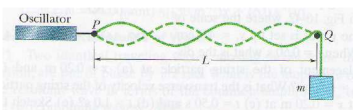
Answer: 20 kg

$$v = \sqrt{\frac{F_T}{\mu}} \rightarrow F_T = mg = \mu v^2 \rightarrow m = \frac{\mu v^2}{g} = \frac{\mu L^2 f_2^2}{g} = \frac{(0.0049kg/m)(2m)^2(100/s)^2}{9.8m/s^2} = 20kg$$

$$f_2 = \frac{v}{\lambda_2} = \frac{2v}{2L} \qquad v^2 = (Lf_2)^2$$

(Second harmonic is n = 2)

Example



Q. In Fig. 16-44, a string, tied to a sinusoidal oscillator at P and running over a support at Q, is stretched by a block of mass m. The separation L between P and Q is 1.20 m, and the frequency f of the oscillator is fixed at 120 Hz. The amplitude of the motion at P is small enough for that point to be considered a node. A node also exists at Q. A standing wave appears when the mass of the hanging block is 286.1 g or 447.0 g, but not for any intermediate mass. What is the linear density of the string?

Example

Two sinusoidal waves, identical except for phase, travel in the same direction along a string and interfere to give a resultant wave

$$y'(x,t) = (\mathbf{3.0\ mm})\sin(20x - 4.0t + 0.820)$$

with x in meters and t in seconds.

What are a) wavelength; b) phase difference; c) amplitude of two original waves?

Example

Compare resultant

$$y'(x,t) = (\mathbf{3.0\ mm})\sin(20x - 4.0t + 0.820)$$

with general form for addition of two waves of same frequency & amplitude:

$$y(x,t) = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{1}{2}\phi\right)$$

Read off:

$$\begin{aligned} 2A\cos(\phi/2) &= 3.0\ \mathrm{mm} \\ \phi/2 &= 0.820 \\ k \equiv 2\pi/\lambda &= 20m^{-1} \\ \omega &= 4.0s^{-1} \end{aligned}$$

which gives $\lambda = 0.31$ m, $\phi = 1.64$ rad, $A = 2.2$ mm