Chapter 8 Torque and Angular Momentum

Review of Chapter 5

We had a table comparing parameters from linear and rotational motion. Today we fill in the table. Here it is

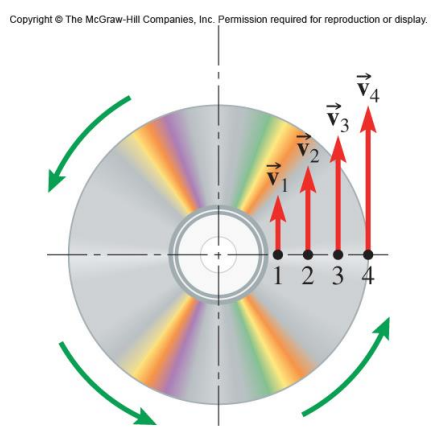
Description	Linear	Rotational
position	x	θ
displacement	Δx	$\Delta\theta$
Rate of change of position	v_x	ω
Average rate of change of position	$v_{x,av} = \frac{\Delta x}{\Delta t}$	$\omega_{av} = \frac{\Delta\theta}{\Delta t}$
Instantaneous rate of change of position	$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$	$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$
Average rate of change of speed	$a_{x,av} = \frac{\Delta v_x}{\Delta t}$	$\alpha_{av} = \frac{\Delta\omega}{\Delta t}$
Instantaneous rate of change of speed	$a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t}$	$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$
Inertia	m	I
Influence that causes acceleration	F	τ
Momentum	$\vec{\mathbf{p}}$	$\vec{\mathbf{L}}$

The relations (often physical laws) for rotational motion are found by a simple substitution of rotational variables for the corresponding linear variables.

Rotational Kinetic energy

A wheel suspended at its axis can spin in space. Since the points of the wheel are moving, the wheel has kinetic energy.

All the pieces in a **rigid body** remain at the same location relative to all the other pieces. For a rotating object, the parts further away from the axis of rotation are moving faster.



$$v = r\omega$$

The total kinetic energy of all the pieces will be

$$K_{total} = \tfrac{1}{2} m_1 v_1^{\,2} + \tfrac{1}{2} m_2 v_2^{\,2} + \cdots + \tfrac{1}{2} m_N v_N^{\,2}$$
$$= \sum_{i=1}^{N} \tfrac{1}{2} m_i v_i^{\,2}$$

$$K_{total} = \sum_{i=1}^{N} \tfrac{1}{2} m_i r_i^2 \omega^2$$
$$= \tfrac{1}{2}\left(\sum_{i=1}^{N} m_i r_i^2\right)\omega^2$$

The quantity in parentheses is called the **rotational inertia** (or the moment of inertia)

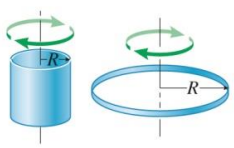
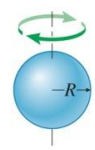
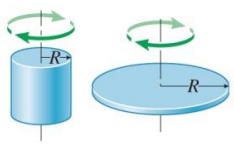
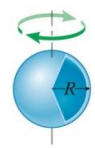
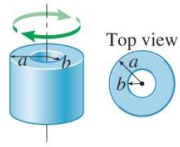
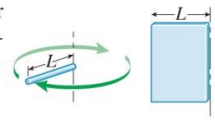
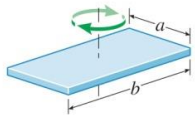
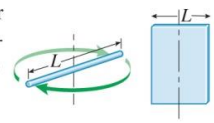
$$I = \sum_{i=1}^{N} m_i r_i^2$$

Finding the Rotational Inertia (page 262)

1. If the object consists of a small number of particles, calculate the sum directly.
2. For symmetrical objects with simple geometric shapes, calculus can be used to perform the sum.
3. Since the rotational inertia is a sum, you can always mentally decompose the object into several parts, find the rotational inertia of each part, and then add them.

The rotational inertia depends on the location of the rotation axis. The same object will have a different rotational inertia depending on where it is rotating. Look at the formula for a thin rod below.

Table 8.1 Rotational Inertia for Uniform Objects with Various Geometrical Shapes

Shape	Axis of Rotation	Rotational Inertia	Shape	Axis of Rotation	Rotational Inertia
Thin hollow cylindrical shell (or hoop) 	Central axis of cylinder	MR^2	Solid sphere 	Through center	$\frac{2}{5}MR^2$
Solid cylinder (or disk) 	Central axis of cylinder	$\frac{1}{2}MR^2$	Thin hollow spherical shell 	Through center	$\frac{2}{3}MR^2$
Hollow cylindrical shell or disk 	Central axis of cylinder	$\frac{1}{2}M(a^2 + b^2)$	Thin rod (or rectangular plate) 	Perpendicular to rod through end (or along edge of plate)	$\frac{1}{3}ML^2$
Rectangular plate 	Perpendicular to plate through center	$\frac{1}{12}M(a^2 + b^2)$	Thin rod (or rectangular plate) 	Perpendicular to rod through center (or parallel to edge of plate through center)	$\frac{1}{12}ML^2$

The rotational kinetic energy of a rigid object rotating with angular velocity ω is

$$K = \tfrac{1}{2} I\omega^2$$

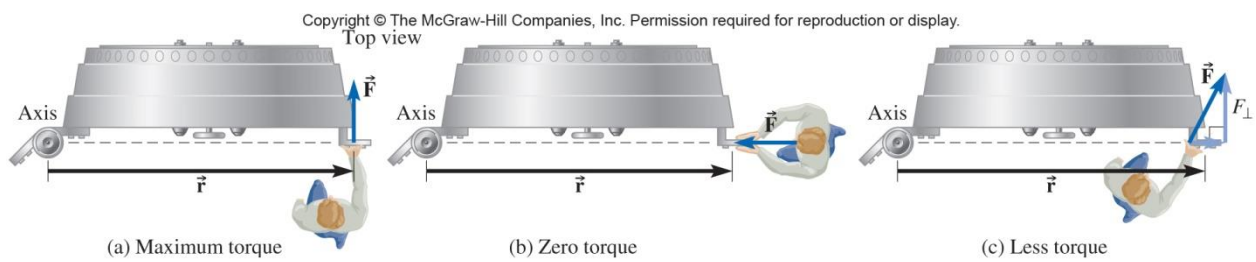
Compare to the translational kinetic energy

$$K = \tfrac{1}{2} mv^2$$

Torque

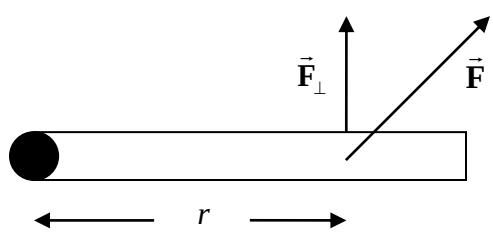
A quantity related to force, called torque, plays the role in rotation that force itself plays in translation. A torque is not separate from a force; it is impossible to exert a torque without exerting a force. Torque is a measure of how effective a given force is at twisting or turning something.

The torque due to a force depends of the magnitude of the applied force, the force's point of application, and the force's direction.



First definition of torque

$$\tau = \pm r F_\perp$$

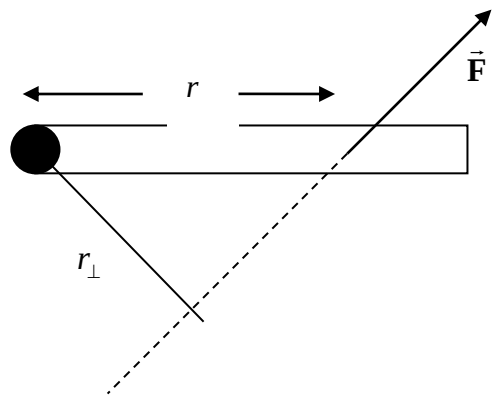


Because rotations have directions, we assign the + sign to torques that cause counterclockwise rotations, and – sign to torques that cause clockwise rotations. What is the sign of the torque in the figure?

Torques are measured in the units of force times distance. This is the same dimensions as work. However, torque has a different effect than work. To keep the two concepts distinct, we measure work in joules and torque in newton-meters.

Second definition of torque

$$\tau = \pm r_\perp F$$

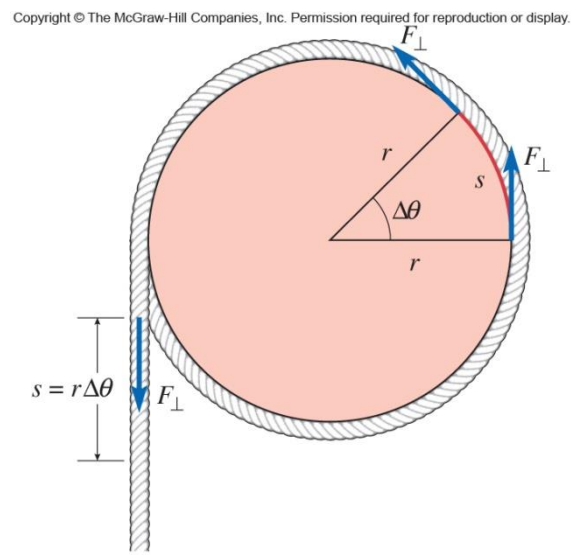


Find the **lever arm** (or **moment arm**) by extending the line of the force and drawing a line from the axis of rotation so that is crosses the line of the force at a right angle. Finding the lever arm is often <u>the most difficult part</u> of a torque problem.

Work done by a torque

The expression for the work done is

$$\begin{aligned} W &= F_{\perp} s \\ &= \left(\frac{\tau}{r}\right)(r\Delta\theta) \\ &= \tau\Delta\theta \end{aligned}$$

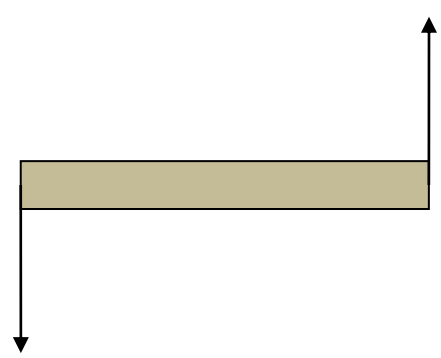


Power is the rate of doing work

$$\begin{aligned} P &= \frac{\Delta W}{\Delta t} \\ &= \frac{\tau\Delta\theta}{\Delta t} \\ &= \tau\omega \end{aligned}$$

Rotational Equilibrium

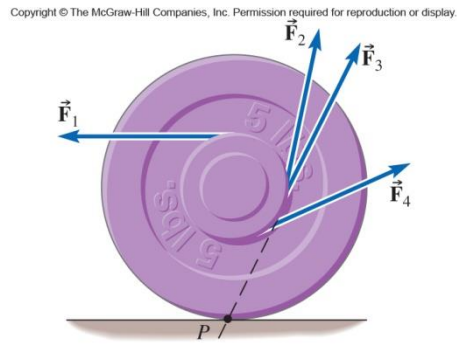
We remember that for an object to remain at rest, the net force acting on it must be equal to zero. (Newton's first law.) However, that condition is not sufficient for rotational equilibrium. What happens to the object to the right?



Conditions for equilibrium (both translational and rotational):

$$\sum \vec{\mathbf{F}} = 0 \text{ and } \sum \tau = 0$$

The obedient spool.
$\mathbf{F}_1$ and $\mathbf{F}_2$ make the spool roll to the left, $\mathbf{F}_4$ to the right, and $\mathbf{F}_3$ makes it slide.

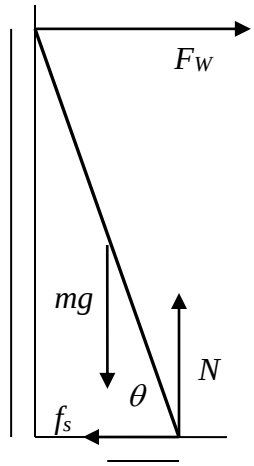


Problem-Solving Steps in Equilibrium Problems (page 274)

1. Identify an object or system in equilibrium. Draw a diagram showing all the forces acting on that object, each drawn at its point of application. Use the center of gravity (CM) as the point of application of any gravitational forces.
2. To apply the force conditions, choose a convenient coordinate system and resolve each force into its *x*- and *y*-components.
3. To apply the torque condition, choose a convenient rotation axis – generally one that passes through the point of application of an unknown force. Then find the torque due to each force. Use whichever method is easier: either the lever arm times the magnitude of the force or the distance times the perpendicular component of the force. Determine the direction of each torque; then either set the sum of all torques (with their algebraic signs) equal to zero or set the magnitude of the CW torques equal to the magnitudes of the CCW torques.
4. Not all problems require all three equilibrium equations (two force component equations and one torque equation). Sometimes it is easier to use more than one torque equation, with a different axis. Before diving in and writing down all the equations, think about which approach is the easiest and most direct.

There are many good examples worked out for you in the text. See pages 274-279.

Example: What is the largest angle a ladder can make so that it does not slide?



We will use the condition for rotational equilibrium

$$\sum \tau = 0$$

We can choose any axis about which to take torques. The axis I choose is where the ladder touches the floor. The lever arms for the normal force and the frictional force will be zero and their torques will also be zero. Recall that the torque is

$$\tau = \pm F r_{\perp}$$

If the ladder has length L, the lever arm for the weight is the short horizontal line below the floor in the diagram. The lever arm is "the perpendicular distance from the line of the force to the point of rotation". Here it is

$$r_{\perp} = \frac{L}{2}\cos\theta$$

The lever arm for the force of the wall pushing against the ladder is

$$r_{\perp} = L\sin\theta$$

Using the condition for rotational equilibrium

$$\begin{aligned} \sum \tau &= 0 \\ \tau_F + \tau_{mg} &= 0 \\ -F_W L\sin\theta + mg\frac{L}{2}\cos\theta &= 0 \end{aligned}$$

The conditions for translational equilibrium are

$$\sum F_x = 0 \text{ and } \sum F_y = 0$$

The *x*-components

$$\begin{aligned} \sum F_x &= 0 \\ F_W - f_s &= 0 \\ F_W &= f_s \end{aligned}$$

The *y*-components

$$\begin{aligned} \sum F_y &= 0 \\ N - mg &= 0 \\ N &= mg \end{aligned}$$

Oh, no! Four equations:

$$\begin{aligned} -F_W L \sin\theta + mg\frac{L}{2}\cos\theta &= 0 \\ F_W &= f_s \\ N &= mg \\ f_s &\le \mu_s N \end{aligned}$$

Use the torque relation

$$\begin{aligned} -F_W L \sin\theta + mg\frac{L}{2}\cos\theta &= 0 \\ F_W L \sin\theta &= mg\frac{L}{2}\cos\theta \\ F_W \sin\theta &= \frac{mg}{2}\cos\theta \\ \tan\theta &= \frac{mg}{2F_W} \end{aligned}$$

Use the two force equations

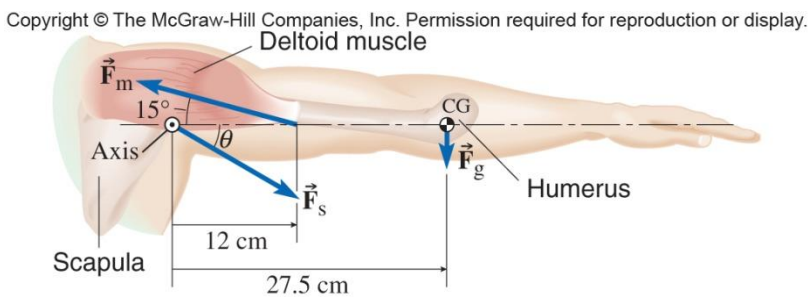
$$\tan\theta = \frac{mg}{2F_W}$$
$$= \frac{N}{2f_s}$$
$$f_s = \frac{N}{2\tan\theta}$$

But

$$f_s \le \mu_s N$$
$$\frac{N}{2\tan\theta} \le \mu_s N$$
$$1 \le 2\mu_s \tan\theta$$
$$\tan\theta \ge \frac{1}{2\mu}$$

If the coefficient of static friction is 0.4, the angle smallest angle is 51°.

Equilibrium in the Human Body
Forces act on the structures in the body.



Example 8.10
The deltoid muscle exerts $\mathbf{F}_m$ on the humerus as shown. The force does two things. The vertical component supports the weight of the arm and the horizontal component stabilizes the joint by pulling the humerus in against the shoulder.

There are three forces acting on the arm: its weight ($\mathbf{F}_g$), the force due so the deltoid muscle ($\mathbf{F}_m$) and the force of the shoulder joint ($\mathbf{F}_s$) constraining the motion of the arm.

Since the arm is in equilibrium, we use the equilibrium conditions. To use the torque equation we use a convenient rotation axis. We choose the shoulder joint as the rotation axis as that will eliminate $\mathbf{F}_s$ from consideration. (Why ?)

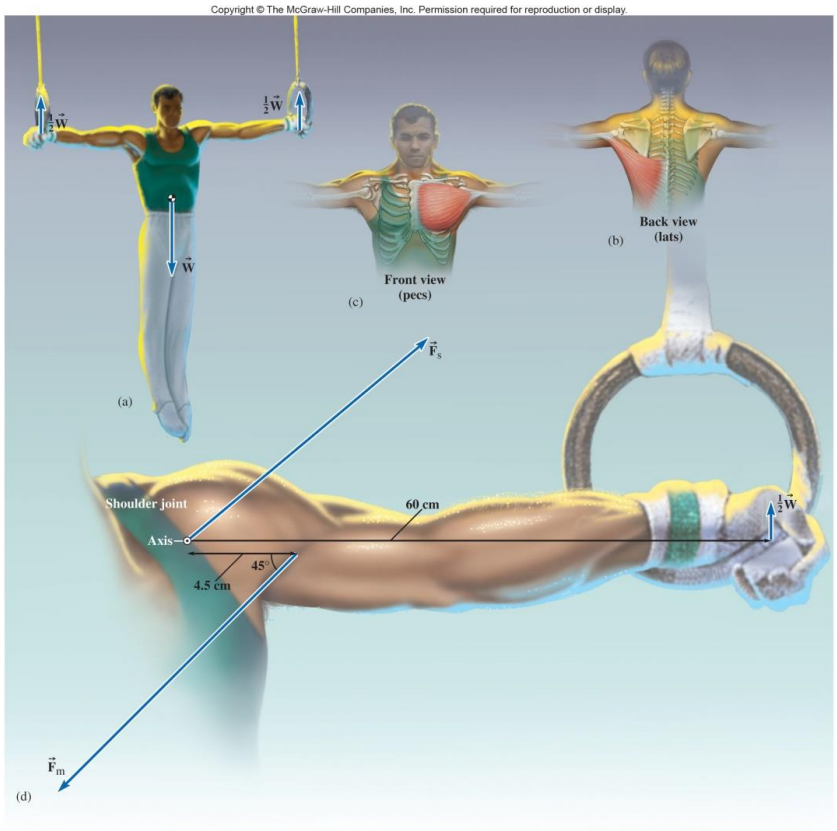
$$\sum \tau = 0$$
$$\tau_g + \tau_m = 0$$
$$-F_g r_g + F_{m\perp} r_m = 0$$
$$-F_g r_g + F_m \sin 15° r_m = 0$$
$$F_m = \frac{F_g r_g}{r_m \sin 15°} = \frac{(30\,\text{N})(0.275\,\text{m})}{(0.12\,\text{m})\sin 15°} = 266\,\text{N}$$

To support the 30 N arm a 270 N force is required. Highly inefficient!!

The Iron Cross.
Here is an interesting video: http://www.youtube.com/watch?v=Sd1LjYgIm_4

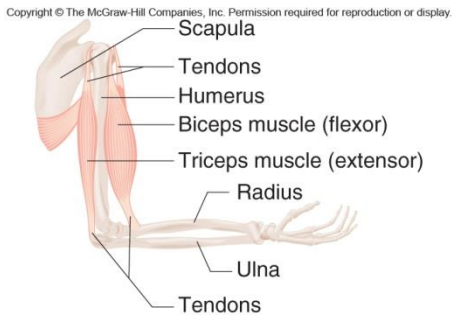
Half of the gymnast's weight is supported by each ring.

$$\sum \tau = 0$$
$$\tau_w + \tau_m = 0$$
$$\tfrac{1}{2} W r_w - F_m r_{\perp m} = 0$$
$$F_m = \frac{W r_w}{2 r_{\perp m}} = \frac{W(0.60\,\text{m})}{2(0.045\,\text{m})\sin 45°} = 9.4W$$



The force exerted by the latissimus dorsi and the pectoralis major on one side of the gymnast's body is more than nine times his weight!

"The structure of the human body makes large muscular forces necessary. Are there any advantages to the structure? Due to the small lever arms, the muscle forces are much larger than they would otherwise be, but the human body has traded this for a wide range of movement of the bones. The biceps and triceps muscles can move the lower arms through almost 180º while they change their lengths by only a few centimeters."



Another video of equilibrium and how easily it can be disrupted.
http://www.youtube.com/watch?v=K6rX1AEi57c

Rotational Form of Newton's second law

$$\sum \tau = I\alpha$$

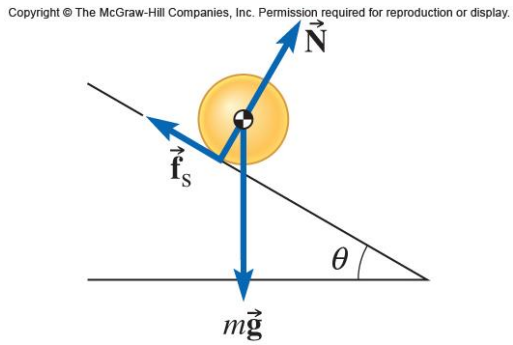
Very similar to other second law.

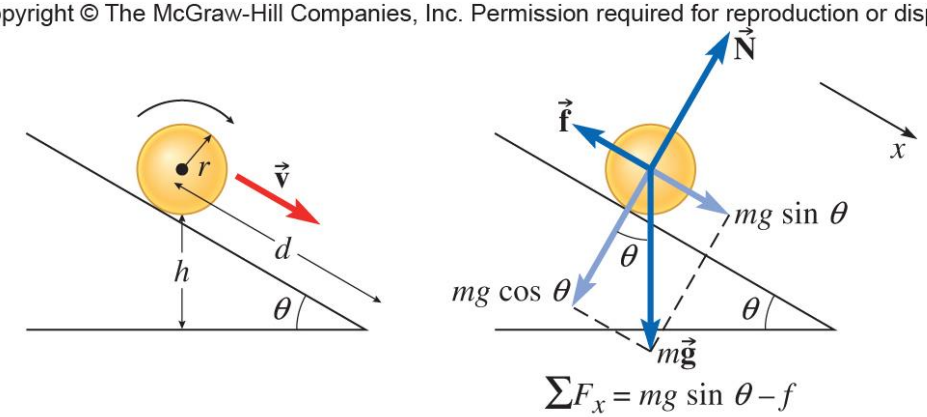
Motion of Rolling Objects

A rolling object has rotational kinetic energy and translational kinetic energy.

$$\begin{aligned} K &= K_{trans} + K_{rot} \\ &= \tfrac{1}{2} m v_{CM}{}^2 + \tfrac{1}{2} I_{CM} \omega^2 \end{aligned}$$

Why does the object roll (and not slide)? Frictional forces exert a torque on the object.





(a) (b)

Example 8.13 The acceleration of a rolling ball.
The rotational form of Newton's second law is

$$\sum \tau = I\alpha$$

The torque on the ball is due to friction

$$\tau = rf$$

So

$$\sum \tau = I\alpha$$
$$rf = I\alpha$$
$$f = \frac{I\alpha}{r}$$

We can use Newton's second law to find the linear acceleration of the ball. As we usually do, take the +*x*-axis is along the incline.

$$\sum F_x = ma_x$$
$$mg\sin\theta - f = ma$$

Use the expression for the frictional force to find,

$$mg\sin\theta - f = ma$$
$$mg\sin\theta - \frac{I\alpha}{r} = ma$$

But the acceleration of the ball is related to its angular acceleration, $a = \alpha r$.

$$mg\sin\theta - \frac{I\alpha}{r} = ma$$
$$mg\sin\theta - \frac{Ia}{r^2} = ma$$
$$mg\sin\theta = \frac{Ia}{r^2} + ma$$
$$a = \frac{mg\sin\theta}{m + I/r^2}$$

For a uniform, solid sphere, $I = (2/5)MR^2$ and for a thin ring, $I = MR^2$. Which has the larger acceleration?

A solid sphere rolls down a hill that has a height h. What is its speed at the bottom?

Use conservation of energy. Since the ball rolls without slipping, the frictional force doesn't do any work. Its displacement is zero in the definition $W = F\,\Delta r\cos\theta$.

$$U_1 + K_1 = U_2 + K_2$$
$$mgy_1 + 0 = 0 + \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

The translational speed of the ball is related to its rotational speed, $v = \omega r$.

$$mgy_1 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$
$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I(v/r)^2$$
$$= \tfrac{1}{2}(m + I/r^2)v^2$$
$$v = \sqrt{\frac{2mgh}{m + I/r^2}}$$

For the solid sphere, $I = (2/5)MR^2$

$$v = \sqrt{\frac{2mgh}{m + I/r^2}} = \sqrt{\frac{2mgh}{m + (2/5)m}} = \sqrt{\frac{2gh}{(7/5)}} = \sqrt{\tfrac{10}{7}gh}$$

This is less than the answer we found when we ignored rolling, $v = \sqrt{2gh}$.

Angular momentum

We introduced the idea of linear momentum in chapter 7. We had

$$\sum \vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}$$

A similar expression exits for rotational motion

$$\sum \tau = \frac{dL}{dt}$$

The net external torque acting on a system is equal to the rate of change of the angular momentum of the system. The angular momentum

$$L = I\omega$$

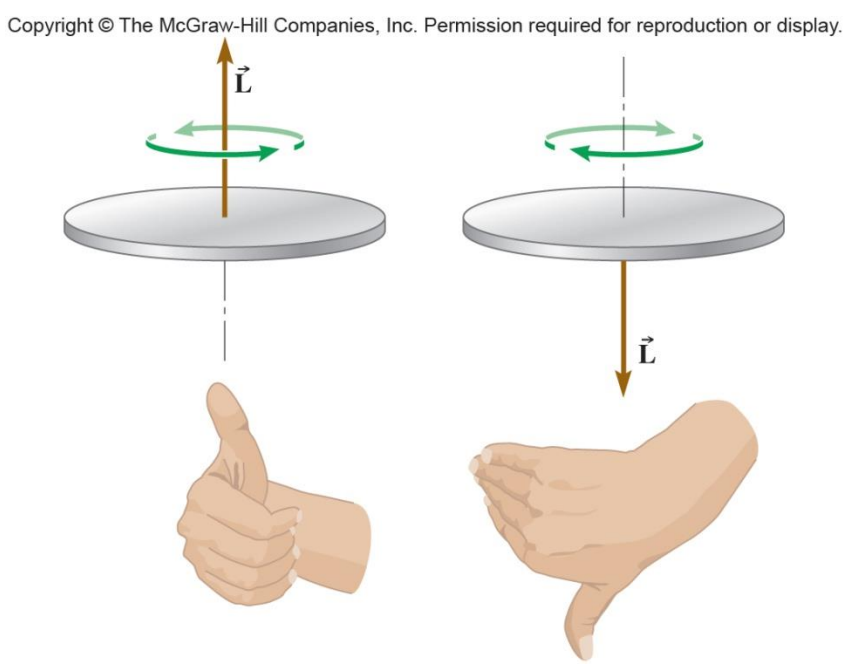
is the tendency of a rotating object to continue rotating with the same angular speed and in the same direction. Angular momentum is measured in kg·m^2/s.

If the net torque is zero, we have the conservation of angular momentum

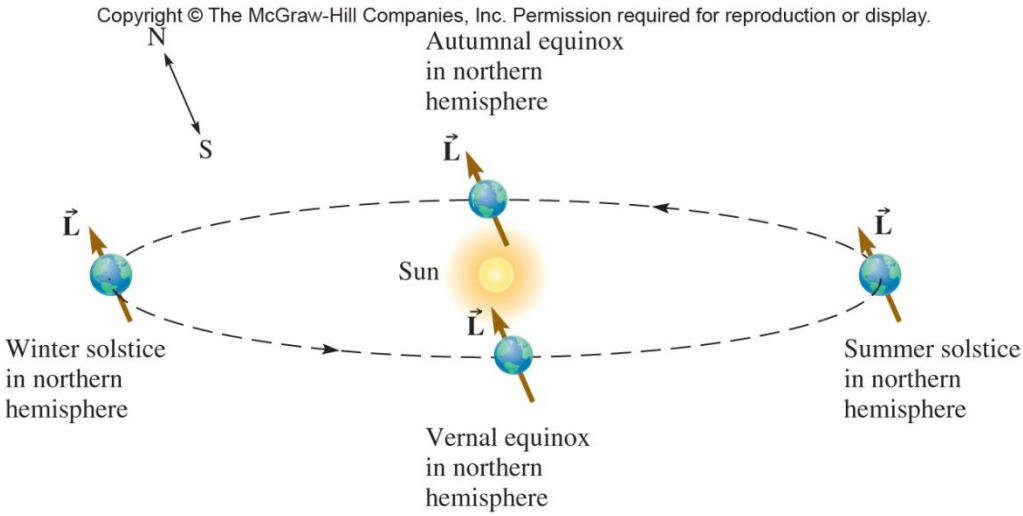
$$\Delta L = 0$$
$$L_i = L_f$$

If the rotational inertia of the system changes, its angular speed will change to compensate.

Angular momentum is a vector. (So is the torque!) The direction is given by the **right-hand rule**.



Our seasons are a consequence of the conservation of angular momentum.



We have completed our study of rotational motion. Try to see how it is analogous to (the more familiar) linear motion. Here is a summary:

Description	Linear	Rotational
position	x	θ
displacement	Δx	$\Delta\theta$
Rate of change of position	v_x	ω
Rate of change of velocity	a_x	α
Average rate of change of position	$v_{x,av} = \frac{\Delta x}{\Delta t}$	$\omega_{av} = \frac{\Delta\theta}{\Delta t}$
Instantaneous rate of change of position	$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$	$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$
Average rate of change of speed	$a_{x,av} = \frac{\Delta v_x}{\Delta t}$	$\alpha_{av} = \frac{\Delta\omega}{\Delta t}$
Instantaneous rate of change of speed	$a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t}$	$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$
Equations of uniform acceleration	$v_{fx} - v_{ix} = a_x \Delta t$ $\Delta x = v_x \Delta t + \frac{1}{2} a_x (\Delta t)^2$ $\Delta x = \frac{1}{2}(v_{fx} + v_{ix})\Delta t$ $v_{fx}^2 - v_{ix}^2 = 2a\Delta x$	$\omega_f - \omega_i = \alpha\Delta t$ $\Delta\theta = \omega\Delta t + \frac{1}{2}\alpha(\Delta t)^2$ $\Delta\theta = \frac{1}{2}(\omega_f + \omega_i)\Delta t$ $\omega_f^2 - \omega_i^2 = 2\alpha\Delta\theta$
Inertia	m	$I = \sum_i m_i r_i^2$
Influence that causes acceleration	$\vec{\mathbf{F}}$	$\tau = F r_\perp = F_\perp r$
Newton's second law	$\sum F_x = m a_x$ $\sum F_y = m a_y$	$\sum \tau = I\alpha$
Work	$W = F\,\Delta r \cos\theta$	$W = \tau\,\Delta\theta$
Kinetic energy	$\frac{1}{2} m v^2$	$\frac{1}{2} I \omega^2$

Momentum	$\vec{\mathbf{p}} = m\vec{\mathbf{v}}$	$\vec{\mathbf{L}} = I\vec{\boldsymbol{\omega}}$
Newton's second law	$\sum \vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}$	$\sum \tau = \frac{dL}{dt}$
Condition for conservation of momentum	$\sum \vec{\mathbf{F}} = 0$	$\sum \tau = 0$
Conservation of momentum	$\vec{\mathbf{p}}_1 = \vec{\mathbf{p}}_2$	$L_1 = L_2$