## Projectile Super Problem

A golf ball is hit from the ground at $35 \mathrm{~m} / \mathrm{s}$ at an angle of $55^{\circ}$. The ground is level.

1. How long is the ball in the air?
2. What is the maximum height of the ball?
3. How far from the launching point does the ball hit the ground?
4. What is the ball's position after 2 seconds. When does it reach this height again?
5. When is the ball 20 m above the ground?

In the plot below, all units are in meters.


## Solution:

1. When the ball hits the ground, $\Delta y=0$.

$$
\begin{aligned}
\Delta y & =v_{i y} \Delta t+\frac{1}{2} a_{y}(\Delta t)^{2} \\
0 & =\Delta t\left(v_{i y}+\frac{1}{2} a_{y} \Delta t\right)
\end{aligned}
$$

Since the only way a product can equal zero is when one of the factors equals zero,

$$
\Delta t=0 \quad \text { or } \quad\left(v_{i y}+\frac{1}{2} a_{y} \Delta t\right)=0
$$

The first condition tells us that the golf ball starts from the ground. The second gives us the time of flight, $\Delta t_{f}$

$$
\begin{aligned}
\left(v_{i y}+\frac{1}{2} a_{y} \Delta t\right) & =0 \\
\frac{1}{2} a_{y} \Delta t & =-v_{i y} \\
\Delta t_{f} & =-\frac{2 v_{i y}}{a_{y}}
\end{aligned}
$$

This is more useful if we use $a_{y}=-g$ and $V_{i y}=V_{i} \sin \theta$ (see the diagram below the trajectory plot).

$$
\begin{aligned}
\Delta t_{f} & =-\frac{2 v_{i y}}{a_{y}} \\
& =\frac{2 v_{i} \sin \theta}{g} \\
& =\frac{2(35 \mathrm{~m} / \mathrm{s}) \sin 55^{\circ}}{9.8 \mathrm{~m} / \mathrm{s}^{2}} \\
& =5.85 \mathrm{~s}
\end{aligned}
$$

What angle maximizes the time of flight? The angle that maximizes $\sin \theta$. The largest sine can be is 1 and that occurs at $90^{\circ}$. Hit the ball straight up!
2. At the maximum height, $v_{f y}=0$.

$$
\begin{aligned}
v_{f y}-v_{i y} & =a_{y} \Delta t \\
0-v_{i} \sin \theta & =-g \Delta t_{h} \\
\Delta t_{h} & =\frac{v_{i} \sin \theta}{g}
\end{aligned}
$$

But this is $1 / 2$ the time of flight. When the ball is shot over level ground half of the time the ball is going up, the other half of the time it is going down. It takes half the total time to reach the highest point.
The height at this time is

$$
\begin{aligned}
\Delta y & =v_{i y} \Delta t+\frac{1}{2} a_{y}(\Delta t)^{2} \\
h & =v_{i} \sin \theta \Delta t_{h}-\frac{1}{2} g\left(\Delta t_{h}\right)^{2} \\
& =v_{i} \sin \theta\left(\frac{v_{i} \sin \theta}{g}\right)-\frac{1}{2} g\left(\frac{v_{i} \sin \theta}{g}\right)^{2}
\end{aligned}
$$

$$
\begin{aligned}
h & =\frac{v_{i}^{2} \sin ^{2} \theta}{g}-\frac{1}{2} g \frac{v_{i}^{2} \sin ^{2} \theta}{g^{2}} \\
& =\frac{v_{i}^{2} \sin ^{2} \theta}{2 g} \\
& =\frac{(35 \mathrm{~m} / \mathrm{s})^{2} \sin ^{2} 55^{\circ}}{2\left(9.8 \mathrm{~m} / \mathrm{s}^{2}\right)} \\
& =41.9 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

What angle maximizes the height? We need to find the maximum of $\sin ^{2} \theta$. The maximum of $\sin ^{2} \theta$ occurs at the maximum of $\sin \theta$, which again is $90^{\circ}$. Hit it straight up.
3. The ball hits the ground when $\Delta t=\Delta t_{f}$. The horizontal position where it hits the ground is called the range ( $R$ ).

$$
\begin{aligned}
\Delta x & =v_{i x} \Delta t \\
R & =v_{i} \cos \theta \Delta t_{f} \\
& =v_{i} \cos \theta \frac{2 v_{i} \sin \theta}{g} \\
& =\frac{v_{i}^{2} 2 \sin \theta \cos \theta}{g}
\end{aligned}
$$

This can be rewritten using the identity $\sin 2 \theta=2 \sin \theta \cos \theta$,

$$
\begin{aligned}
R & =\frac{v_{i}^{2} \sin 2 \theta}{g} \\
& =\frac{(35 \mathrm{~m} / \mathrm{s})^{2} \sin \left[2\left(55^{\circ}\right)\right]}{9.8 \mathrm{~m} / \mathrm{s}^{2}} \\
& =117 \mathrm{~m}
\end{aligned}
$$

What angle maximizes the range? This time we want to find the maximum of $\sin 2 \theta$. The maximum of sine occurs at $90^{\circ}$. This time $2 \theta=90^{\circ}$ or $\theta=45^{\circ}$. The angle required for maximum range over level ground is $45^{\circ}$. Also the range is symmetric about $45^{\circ}$. For some angle $\alpha$,

$$
\sin \left[2\left(45^{\circ}+\alpha\right)\right]=\sin \left(90^{\circ}+2 \alpha\right)=\cos 2 \alpha
$$

and

$$
\sin \left[2\left(45^{\circ}-\alpha\right)\right]=\sin \left(90^{\circ}-2 \alpha\right)=\cos (-2 \alpha)=\cos 2 \alpha
$$

The range for $\theta_{1}=45^{\circ}+\alpha$ is the same for $\theta_{2}=45^{\circ}-\alpha$. Another way is to say if

$$
\theta_{1}+\theta_{2}=\left(45^{\circ}+\alpha\right)+\left(45^{\circ}-\alpha\right)=90^{\circ}
$$

the ranges are the same!

4. Where is the ball at 2 seconds? Its horizontal position is found using

$$
\begin{aligned}
\Delta x & =v_{i x} \Delta t \\
& =v_{i} \cos \theta \Delta t_{f} \\
& =(35 \mathrm{~m} / \mathrm{s}) \cos 55^{\circ}(2 \mathrm{~s}) \\
& =40.2 \mathrm{~m}
\end{aligned}
$$

Its vertical position is

$$
\begin{aligned}
\Delta y & =v_{i y} \Delta t+\frac{1}{2} a_{y}(\Delta t)^{2} \\
& =v_{i} \sin \theta \Delta t-\frac{1}{2} g(\Delta t)^{2} \\
& =(35 \mathrm{~m} / \mathrm{s}) \sin 55^{\circ}(2 \mathrm{~s})-\frac{1}{2}\left(9.8 \mathrm{~m} / \mathrm{s}^{2}\right)(2 \mathrm{~s})^{2} \\
& =37.7 \mathrm{~m}
\end{aligned}
$$

When is it at this height again? There are many ways to find this. First, note that the time of flight of the ball is 5.85 s from part 1 . Since the trajectory is symmetric, if it reaches this height 2 s after launch, it will reach it again 2 s before it lands,

$$
\Delta t=5.85 \mathrm{~s}-2.00 \mathrm{~s}=3.85 \mathrm{~s}
$$

Another way is to use the symmetry of the $y$-component of the velocity. The $y$ components of the velocities at the same heights have the same magnitudes but opposite signs. At 2 s

$$
\begin{aligned}
v_{f y}-v_{i y} & =a_{y} \Delta t \\
v_{f y} & =v_{i} \sin \theta-g \Delta t \\
& =(35 \mathrm{~m} / \mathrm{s}) \sin 55^{\circ}-\left(9.8 \mathrm{~m} / \mathrm{s}^{2}\right)(2 \mathrm{~s}) \\
& =9.07 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

When is the speed $-9.07 \mathrm{~m} / \mathrm{s}$ ?

$$
\begin{aligned}
v_{f y}-v_{i y} & =a_{y} \Delta t \\
v_{f y}-v_{i} \sin \theta & =-g \Delta t \\
\Delta t & =\frac{v_{f y}-v_{i} \sin \theta}{-g} \\
& =\frac{-9.07 \mathrm{~m} / \mathrm{s}-(35 \mathrm{~m} / \mathrm{s}) \sin 55^{\circ}}{-9.8 \mathrm{~m} / \mathrm{s}^{2}} \\
& =3.85 \mathrm{~s}
\end{aligned}
$$

Finally, the worst way is to just solve for $\Delta t$ using the quadratic formula,

$$
\begin{aligned}
\Delta y & =v_{i y} \Delta t+\frac{1}{2} a_{y}(\Delta t)^{2} \\
& =v_{i} \sin \theta \Delta t-\frac{1}{2} g(\Delta t)^{2} \\
37.7 \mathrm{~m} & =(35 \mathrm{~m} / \mathrm{s}) \sin 55^{\circ} \Delta t-\frac{1}{2}\left(9.8 \mathrm{~m} / \mathrm{s}^{2}\right)(\Delta t)^{2} \\
\frac{1}{2}\left(9.8 \mathrm{~m} / \mathrm{s}^{2}\right)(\Delta t)^{2}-(35 \mathrm{~m} / \mathrm{s}) \sin 55^{\circ} \Delta t+37.7 \mathrm{~m} & =0 \\
\left(4.9 \mathrm{~m} / \mathrm{s}^{2}\right)(\Delta t)^{2}-(28.67 \mathrm{~m} / \mathrm{s}) \Delta t+37.7 \mathrm{~m} & =0 \\
\Delta t & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& =\frac{-(-28.67) \pm \sqrt{(28.67)^{2}-4(4.9)(37.7)}}{2(4.9)} \\
& =\frac{28.67 \pm 9.11}{9.8} \\
& =2.00 \mathrm{~s}, 3.86 \mathrm{~s}
\end{aligned}
$$

The first answer is then it reaches 37.7 m while ascending (we knew it would be 2 s ), the second is when it reaches 37.7 m while descending. (I quit writing units in the quadratic because it makes the equation even more unwieldy.)
5. To find when the ball reaches 20 m we use the quadratic equation again,

$$
\begin{aligned}
\Delta y & =v_{i y} \Delta t+\frac{1}{2} a_{y}(\Delta t)^{2} \\
& =v_{i} \sin \theta \Delta t-\frac{1}{2} g(\Delta t)^{2} \\
20 \mathrm{~m} & =(35 \mathrm{~m} / \mathrm{s}) \sin 55^{\circ} \Delta t-\frac{1}{2}\left(9.8 \mathrm{~m} / \mathrm{s}^{2}\right)(\Delta t)^{2} \\
\frac{1}{2}\left(9.8 \mathrm{~m} / \mathrm{s}^{2}\right)(\Delta t)^{2}-(35 \mathrm{~m} / \mathrm{s}) \sin 55^{\circ} \Delta t+20 \mathrm{~m} & =0 \\
\left(4.9 \mathrm{~m} / \mathrm{s}^{2}\right)(\Delta t)^{2}-(28.67 \mathrm{~m} / \mathrm{s}) \Delta t+20 \mathrm{~m} & =0 \\
\Delta t & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& =\frac{-(-28.67) \pm \sqrt{(28.67)^{2}-4(4.9)(20)}}{2(4.9)} \\
& =\frac{28.67 \pm 20.74}{9.8} \\
& =0.81 \mathrm{~s}, 5.04 \mathrm{~s}
\end{aligned}
$$

Notice that the sum of these times is 5.85 s , the time of flight.
Summary: Derived equations for a projectile launched from level ground with initial velocity $v_{i}$ at an angle $\theta$ above the ground:

$$
\begin{aligned}
\text { Time of flight } & \Delta t_{f}=\frac{2 v_{i} \sin \theta}{g} \\
\text { Time to height } & \Delta t_{h}=\frac{v_{i} \sin \theta}{g}=\frac{1}{2} \Delta t_{f} \\
\text { Maximum height } & h=\frac{v_{i}^{2} \sin ^{2} \theta}{2 g} \\
\text { Range } & R=\frac{v_{i}^{2} \sin 2 \theta}{g}
\end{aligned}
$$

If the ground is not level, for example throwing a ball from the top of a building, these equations will not apply (unless the initial and final heights are the same)!

The projectile travels in a parabolic path as long as we neglect air resistance. The motion is symmetric about the maximum height (the vertex of the parabola).

