Problem 1: James and Mary are buying pizza (yay!)

Two coupons:
(i) 12 in. dia. for \$9.00
(ii) 16 in. dia. for \$12.00

(a) We may treat each pizza as a circular disk, with area $A_{pizza} = \pi\left(\frac{d}{2}\right)^2$, where "d" is the diameter of the pizza. To find the area in [m²], we first convert inches to cm, then cm to m:

$$\text{(i)} \quad 12 \text{ in} \left(\frac{2.54 \text{ cm}}{1 \text{ in}}\right)\left(\frac{1 \text{ m}}{100 \text{ cm}}\right) = 0.3048 \text{ m}$$

Thus $A_{\substack{12 \text{ in} \\ \text{pizza}}} = \pi\left(\frac{0.3048}{2}\right)^2 = \boxed{0.073 \text{ m}^2}$

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$$\text{(ii)} \quad 16 \text{ in} \left(\frac{2.54 \text{ cm}}{1 \text{ in}}\right)\left(\frac{1 \text{ m}}{100 \text{ cm}}\right) = 0.4064 \text{ m}$$

Thus $A_{\substack{16 \text{ in.} \\ \text{pizza}}} = \pi\left(\frac{0.4064}{2}\right)^2 = \boxed{0.13 \text{ m}^2}$

b) Here, we calculate \$/Area for both pizzas. This will tell us how much each pizza costs per unit area; the lower the value, the better the deal.

$$\text{(i)} \quad \frac{\$}{A} = \frac{\$9.00}{A_{\substack{12 \text{ in} \\ \text{pizza}}}} = \frac{\$9.00}{0.073 \text{ m}^2} = 123 \ \$/\text{m}^2,$$

$$\text{(ii)} \quad \frac{\$}{A} = \frac{\$12.00}{A_{\substack{16 \text{ in} \\ \text{pizza}}}} = \frac{\$12.00}{0.13 \text{ m}^2} = 92.3 \ \$/\text{m}^2.$$

Deal (i) gives a 1 m² pizza for \$123, whereas deal (ii) gives a 1 m² pizza for \$92. Deal (ii) gives us more pizza for our money, and is thus "better."

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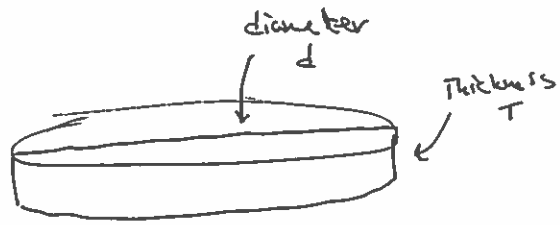
(c) Since the thicknesses of the pizzas are different, the area no longer measures "how much pizza we have" accurately for a comparison. We instead must consider the volumes of each pizza:

Assume a right-cylinder pizza,



$$\text{Volume} = \pi\left(\frac{d}{2}\right)T$$

Then the relevant ratio is $\$/m^3$ ← volume:

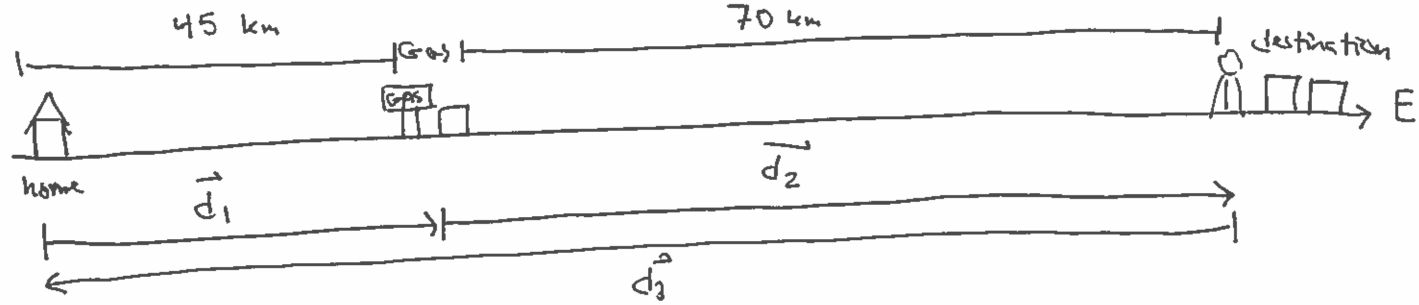
(i) $$\frac{\$}{m^3} = \frac{\$9.00}{A_{12\text{ in. pizza}} \cdot \text{thickness}} = \frac{\$9.00}{(0.073\ m^2)(0.0254\ m)} = \frac{\$4850}{m^3}$$

↑ 1 in. thick

(ii) $$\frac{\$}{m^3} = \frac{\$12.00}{A_{16\text{ in pizza}} \cdot \text{thickness}} = \frac{\$12.00}{(0.13)(0.7)(0.0254)} = \frac{\$5190}{m^3}$$

Here we see that deal (i) gives us more volume of pizza for our dollar, and is a better deal.

Problem 2: We may draw Becky's trip along a line



The trip is really the sum of three displacements:

$\vec{d_1} = 45$ km E from home to Gas

$\vec{d_2} = 70$ km E from gas to destination

$\vec{d_3} = -(\vec{d_1} + \vec{d_2})$ from destination to home.

(a) From the picture, we see that this is just

$$\vec{d_2} + \vec{d_3} = \vec{d_2} + (-\vec{d_1} - \vec{d_2}) = -\vec{d_1}$$

or $\boxed{45 \text{ km West.}}$

(b) $\vec{d}_{tot} = \vec{d_1} + \vec{d_2} + \vec{d_3} = \vec{d_1} + \vec{d_2} - (\vec{d_1} + \vec{d_2})$

$\boxed{= \vec{0}.}$ This makes sense, as she begins and ends in the same location.

For the total distance we simply double the distance from home to the destination:

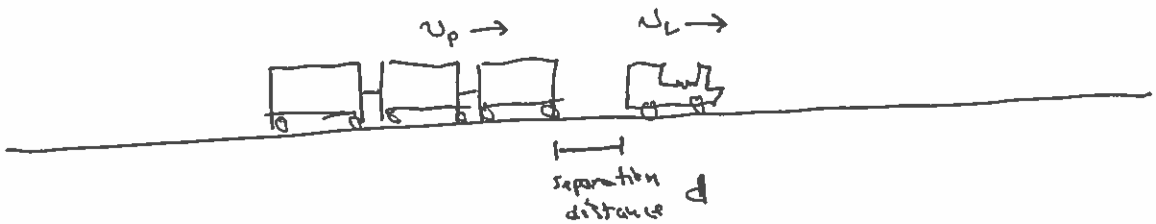
$$d_{tot} = 2(45 + 70) \text{ km}$$

$$\boxed{= 230 \text{ km}}$$

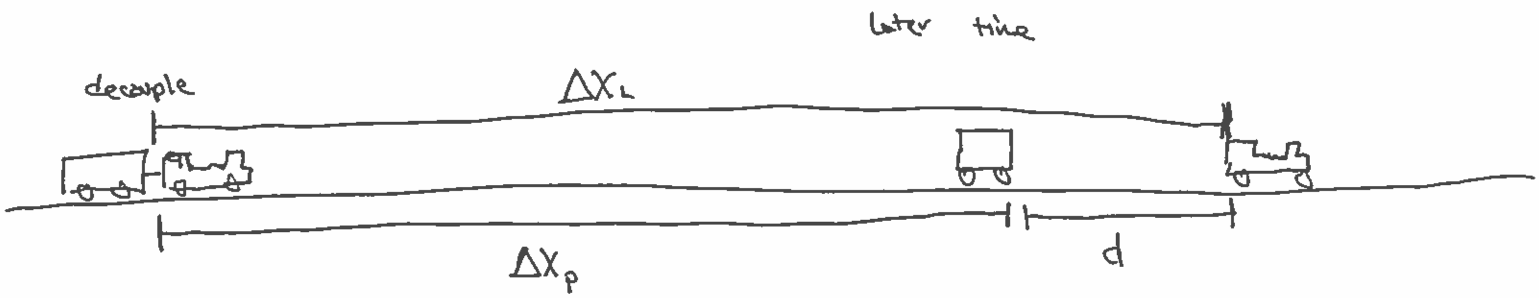
Problem 3: Start with a picture:

$v_L = 20 \, m/s$

$v_p = 13 \, m/s$



When the passenger cars decouple from the locomotive, both are in roughly the same position:



So we may describe the separation distance as the difference in the two displacements:

$$d = \Delta x_L - \Delta x_p.$$

As both the cars and locomotive move at const. speed:

$$\Delta x_L = v_L \Delta t, \quad \Delta x_p = v_p \Delta t,$$

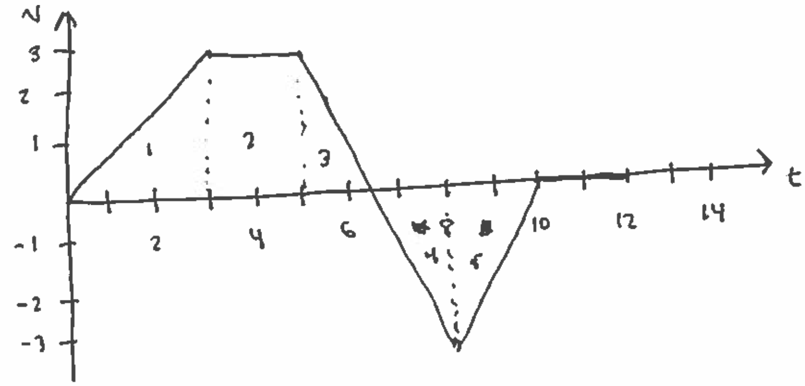
So we see

$$d = v_L \Delta t - v_p \Delta t = (v_L - v_p) \Delta t.$$

To find the time when the separation distance is 350 m, we solve for Δt and use $d = 350 \, m$:

$$\Delta t = \frac{350 \, m}{(20 - 13) \, m/s} = \frac{350 \, m}{7 \, m/s} = \boxed{50 \, s}$$

Problem 4:



(a) $0s < t \le 3s$: The cart begins at rest and travels in the positive x-dir. with constant acceleration (positive)

$3s < t \le 5s$: The cart moves with constant velocity in positive x-dir.

$5s < t \le 8s$: The cart accelerates in the negative x-dir. It slows down (while still moving in pos. x-dir.), stops for an instant, and then speeds up in neg. x-dir.

(b) Recall that displacement is area under velocity vs. time graph. We may break the graph up into 5 simple shapes:

$$\underset{0<t\le 8s}{\text{displacement}} = A_1 + A_2 + A_3 + A_4$$

$$A_1 = \frac{1}{2}(3s)(3\,m/s) = 4.5\,m$$

$$A_2 = (3\,m/s)(2s) = 6\,m$$

$$A_3 = \frac{1}{2}(1.5s)(3\,m/s) = \frac{9}{4}\,m = 2.25\,m$$

$$A_4 = \frac{1}{2}(1.5s)(-3\,m/s) = -\frac{9}{4} = -2.25\,m$$

so $\boxed{\text{disp} = +10.5\,m}$

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(c) We may use the same technique as in (b), but all areas are now positive:

$$\text{tot. distance} = A_1 + A_2 + A_3 + |A_4| + |A_5|$$

$A_1 - A_4$ from (b)

$$A_5 = \frac{1}{2}(2\,s)(-3\,m/s) = -3\,m$$

$$\text{tot. dist.} = (4.5 + 6 + 2.25 + 2.25 + 3)\,m$$

$$\boxed{= 18\,m}$$

(d) The average speed is given by the $\frac{\text{total distance}}{\text{total time}}$. ← from (c) ← 12 s.

Thus $\boxed{\text{av. speed} = \frac{18\,m}{12\,s} = 1.5\,m/s}$

Avg. velocity is $\frac{\text{tot. displacement}}{\text{tot. time}}$. To get the tot. disp., we look to our answer from (b) while including A_5 (with sign):

$$\text{tot. disp} = 10.5\,m - 3\,m$$
$$= +7.5\,m$$

then $\boxed{\vec{v}_{av} = \frac{+7.5\,m}{12\,s} = +0.625\,m/s}$

(e) The avg. acceleration is given by $\frac{\Delta V}{\Delta t}$: here, both v_i and v_f are zero! so

$$a_{av} = \frac{0 - 0\,m/s}{12\,s} = 0\,m/s^2.$$

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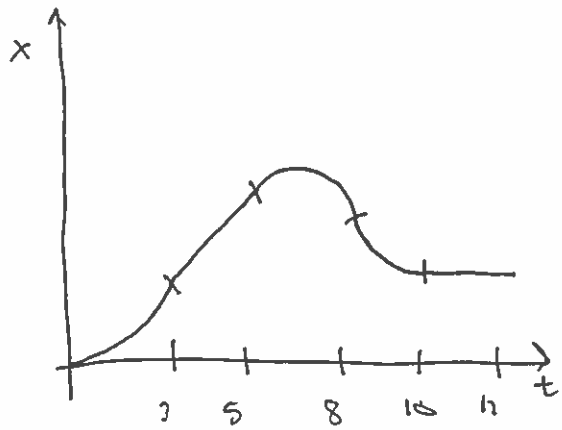
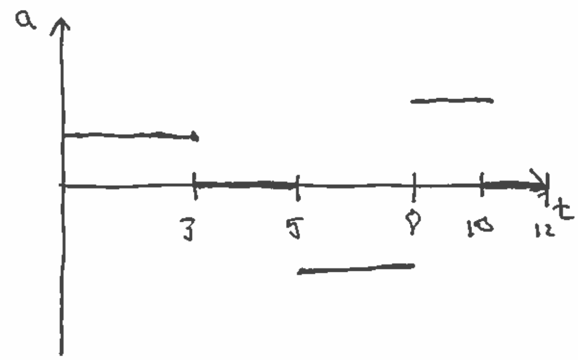
(f) Keyword "sketch" means these don't have to be precise:





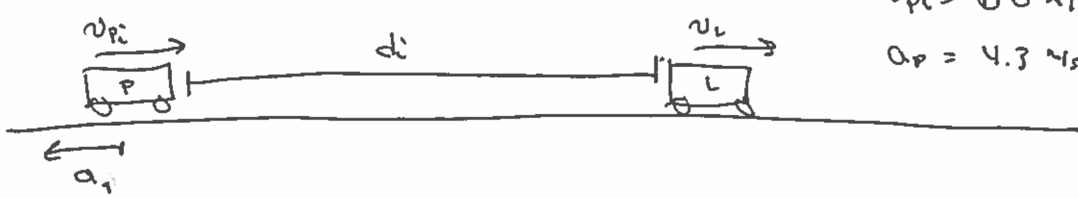
Problem 5: Start with a picture

$v_L = \text{const.} = 25\,m/s$

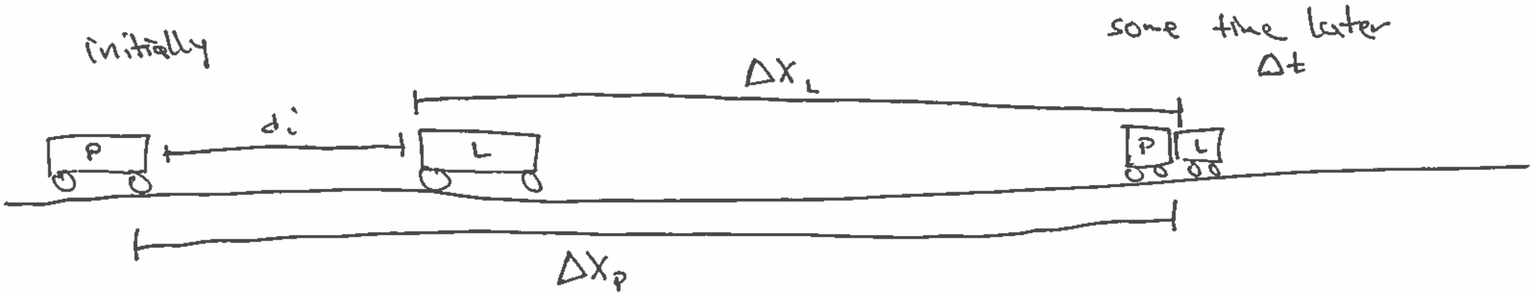
$v_{pi} = 60\,m/s$

$a_p = 4.3\,m/s^2$

$d_i = 100\,m$



Let us assume that the two trains collide. Thus they will exist in the same place at the same time (and have $v_p > v_L$ at this time). At what time does this occur?



We see that for them to be "at the same place" we must have

$$\Delta X_p = d_i + \Delta X_L \qquad (1)$$

We may write ΔX_p and ΔX_L as functions of time Δt through kinematics

(2) $\Delta X_L = v_L \Delta t$ as $v_L = \text{const.}$

(3) $\Delta X_p = v_{pi} \Delta t - \frac{1}{2} a_p (\Delta t)^2$ here the "$-$" sign comes from v_{pi} and a_p pointing in opposite directions.

Thus we plug (2) & (3) into (1) to see

$$v_{pi} \Delta t - \frac{1}{2} a_p (\Delta t)^2 = d_i + v_L \Delta t$$

rearrange to get quadratic eq.

$$\rightarrow \underbrace{\frac{1}{2} a_p}_{a} (\Delta t)^2 + \underbrace{(v_L - v_{pi})}_{b} \Delta t + \underbrace{d_i}_{c} = 0.$$

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Using the quadratic eq.

$$\Delta t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we get

$$\Delta t = \frac{-(v_L - v_{pi}) \pm \sqrt{(v_L - v_{pi})^2 - 4\left(\frac{1}{2}a_p\right)d_i}}{2\left(\frac{1}{2}a_p\right)}$$

$$= \frac{(v_{pi} - v_L) \pm \sqrt{(v_L - v_{pi})^2 - 2a_p d_i}}{a_p}$$

Plugging in numbers gives us

uh oh!!

$$\Delta t = \frac{35\,m/s \pm \sqrt{1225 - 1376}}{4.3}$$

$$\sqrt{1225 - 1376} = \sqrt{-151}$$ this is the sq. root of a negative number!!

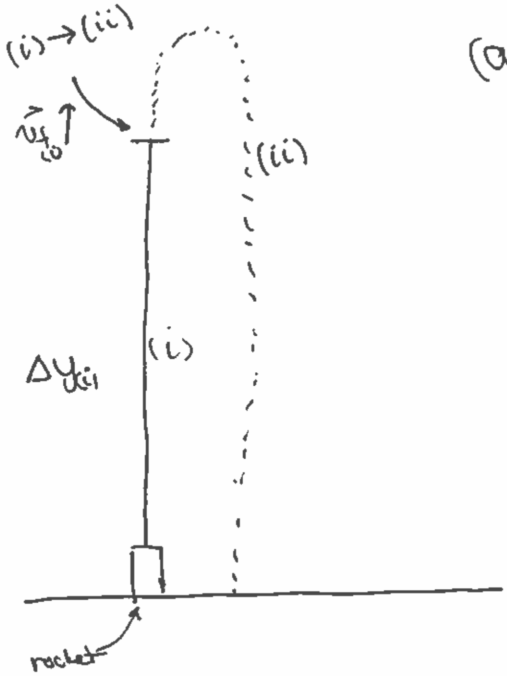
Time is a real, physical thing! It can't be imaginary! So what does this weird answer tell us? Well, we solved for the time at which both trains were at the same position. Since our time is imaginary, it doesn't exist.

Thus the two trains are never at the same position at the same time, so they don't collide.

Problem 6: It is easiest to consider the motion of the rocket in two parts:

(i) constant upward accel. $a_r = 6.3 \, m/s^2$

(ii) const. downward accel. g.



(a) We know that the rocket spends 10 s accel. upwards in part (i). Since it was launched from rest, $v_{i(i)} = 0$. Thus, the final speed after 10 s is given by

$$\boxed{\vec{v}_{f(i)} = \vec{a}_r \, (10 s) = 63 \, m/s \text{ up}}$$

We can also find how high the rocket is off the ground after 10 s:

$$\underline{\Delta y_{(i)} = \frac{1}{2} a_r \, (10 s)^2 = 315 m.}$$ This will be helpful in part (b).

(b) It is at this point we consider (ii) to be an independent problem. Imagine the rocket is an object launched upward with some initial velocity at some height above the ground, and is in freefall.

Thus, it has some initial velocity

$$\vec{v}_{i(ii)} = 63 \, m/s \text{ up}.$$ (the speed of the rocket when its engine stops)

What is the max height of this object? Well, max height is defined when

$$\vec{v} = 0.$$

Using $\vec{v} = \vec{v}_{i(ii)} - g \Delta t$, we see that it attains max height at time

$$\Delta t_{max} = \frac{v_{i(ii)}}{g} = 6.3 s.$$

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This gives us a height

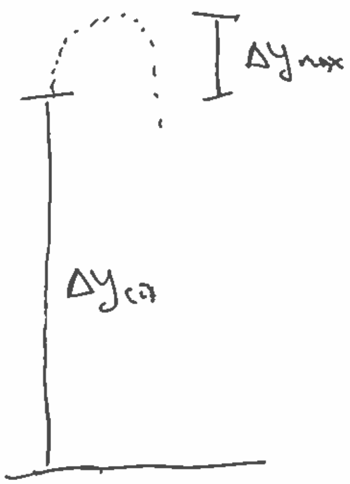
$$\Delta y_{max} = v_{i(ii)} \Delta t_{max} - \frac{1}{2} g (\Delta t_{max})^2$$

$$= (63 \, m/s)(6.3 s) - \frac{1}{2} 10 \, m/s^2 (6.3 s)^2$$

$$= 198.45 \, m.$$

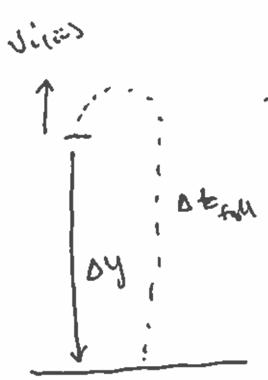
__But__ this is the rocket's max height __above where the engine stopped__

Its true max. height above the ground must also include how high the rocket traveled while the engine was burning:



$$\boxed{\Delta y_{\substack{\text{max above} \\ \text{ground}}} = \Delta y_{max} + \Delta y_{(i)} = 513.45 \, m}$$

(C) Here we may calculate the total air time by first (again) considering problem (ii):



Initial upward velocity $\vec{v}_{i(ii)} = 63 \, m/s$ up in free-fall.

The time it takes to hit the ground is the time it takes to fall $\Delta y = -\Delta y_{(i)}$ (the height it attained when the engine burned out)

$$\longrightarrow -315 m = (63 \, m/s) \Delta t_{fall} - \frac{1}{2} g (\Delta t_{fall})^2$$

Here we must solve the quadratic eqn:

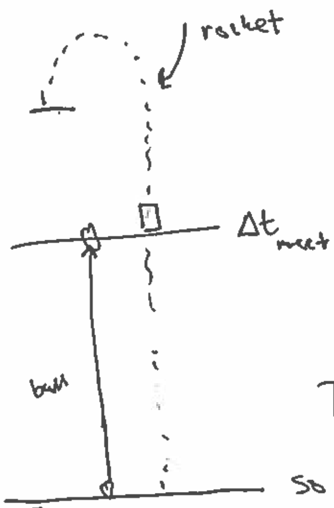
$$\Delta t_{fall} = \frac{63 \, m/s \pm \sqrt{(63)^2 - 4(\frac{1}{2}g)(-315)}}{g} = 16.43 s$$

next page.

This gives us the time it takes the rocket to fall back to Earth, but this is not the total airtime; for that we must also consider the 10 s it spent accelerating upward.

$$\boxed{\begin{aligned}\Delta t_{tot} &= \Delta t_{fall} + 10\,s \\ &= 26.43\,s\end{aligned}}$$

d) This part requires us to find the height of the collision betw the ball and falling rocket. For them to collide, we must have



height of rocket = height of ball

at the same time Δt_{meet}!

The initial height of the rocket is $\Delta y_{(i)} = 315\,m$ for prob (ii), so this becomes

$$\Delta y_{rocket} + 315\,m = \Delta y_{ball}.$$

We must solve this for Δt_{meet}.

From kinematics:

$$\Delta y_{ball} = (85\,m/s)\Delta t - \tfrac{1}{2}g(\Delta t)^2 \quad ; \quad \Delta y_{rocket} = (63\,m/s)\Delta t - \tfrac{1}{2}g(\Delta t)^2$$

$$\rightarrow \quad 63\,\Delta t_{meet} \cancel{- \tfrac{1}{2}g(\Delta t_{meet})^2} + 315\,m = 85\,\Delta t_{meet} \cancel{- \tfrac{1}{2}g(\Delta t_{meet})^2}$$

$$63\,\Delta t_{meet} + 315\,m = 85\,\Delta t_{meet}$$

$$\text{or} \quad \Delta t_{meet} = \frac{315\,m}{85-63} = 14.3\,s.$$

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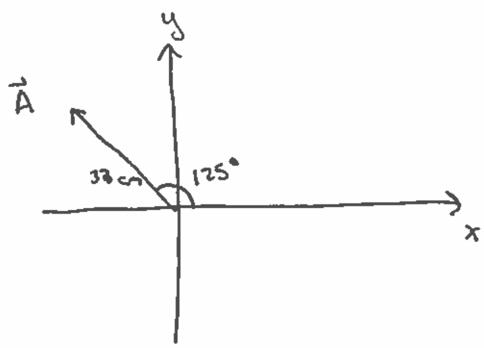
To find the height, we plug this time into e.g.

$$\Delta y_{ball} = (85\,m/s)(14.3s) - \frac{1}{2} g\,(14.3s)^2$$

$$= 193.05\,m$$

So they collide $\boxed{193\,m}$ above the ground.

Problem 7: $\vec{A} = 33\text{ cm at } 125°$



From the formulae for vector components:

$$A_x = A\cos\theta = 33\text{ cm}\cos(125°)$$

$$\boxed{A_x = -18.9\text{ cm}}$$

$$A_y = A\sin\theta = 33\text{ cm}\sin(125°)$$

$$\boxed{A_y = 27.0\text{ cm}}$$

__Problem 8__: Given $\vec{A} = 2\hat{x} - 7\hat{y}$, $\vec{B} = 6\hat{x} + 3\hat{y}$, we find

(a) x- and y- comps. of $\vec{A} - \vec{D}$. Recall that for any two vectors

$$\vec{C} = C_x \hat{x} + C_y \hat{y}$$
$$\vec{D} = D_x \hat{x} + D_y \hat{y}$$

we may write

$$\vec{C} + \vec{D} = (C_x \hat{x} + C_y \hat{y}) + (D_x \hat{x} + D_y \hat{y})$$
$$\underline{\vec{C} + \vec{D} = (C_x + D_x)\hat{x} + (C_y + D_y)\hat{y}}$$

Using this info with $\vec{C} = \vec{A}$ and $\vec{D} = -\vec{B}$, we get

$$\vec{A} - \vec{B} = (A_x - B_x)\hat{x} + (A_y - B_y)\hat{y}$$
$$= (-4)\,\hat{x} + (-10)\,\hat{y}$$

thus $\boxed{(\vec{A} - \vec{B})_x = -4, \quad (\vec{A} - \vec{B})_y = -10}$

b) Now we use $\vec{C} = 2\vec{A}$, $\vec{D} = 3\vec{B}$;

$$2\vec{A} + 3\vec{B} = (2A_x + 3B_x)\hat{x} + (2A_y + 3B_y)\hat{y}$$
$$= (4 + 18)\hat{x} + (-14 + 9)\hat{y}$$
$$= 22\hat{x} - 5\hat{y}.$$

$$\boxed{(2\vec{A} + 3\vec{B})_x = 22, \quad (2\vec{A} + 3\vec{B})_y = -5}$$

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(C) We again start with two vectors

$$(\vec{C}+\vec{D}) = (C_x + D_x)\hat{x} + (C_y + D_y)\vec{y}.$$

To find the magnitude, we square the x- and y-comps.:

$$|\vec{C}+\vec{D}| = \sqrt{(C_x+D_x)^2 + (C_y+D_y)^2}.$$

To find direction use arctangent:

$$\theta_{\vec{C}+\vec{D}} = \arctan\left(\frac{C_y + D_y}{C_x + D_x}\right).$$

Here, $\vec{C} = \vec{A}$ and $\vec{D} = \vec{B}$, so

$$|\vec{A}+\vec{B}| = \sqrt{(A_x+B_x)^2 + (A_y+B_y)^2}$$

$$= \sqrt{(8)^2 + (-4)^2}$$

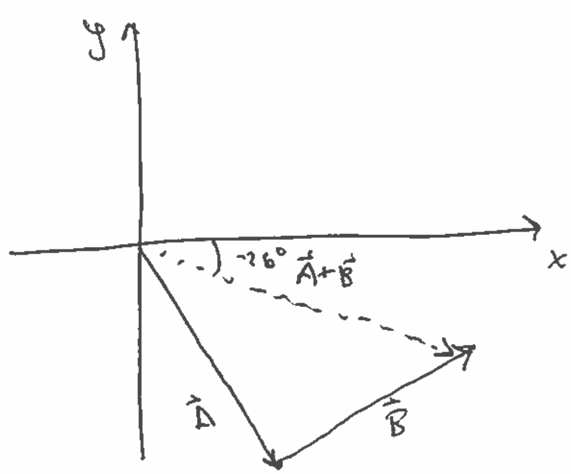
$|\vec{A}+\vec{B}| = 8.94$

$$\theta_{\vec{A}+\vec{B}} = \arctan\left(\frac{A_y+B_y}{A_x+B_x}\right)$$

$$= \arctan\left(\frac{-4}{8}\right)$$

$\theta_{\vec{A}+\vec{B}} = -26.56°$ or $333.4°$

To check, sketch $\vec{A}+\vec{B}$



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(d) Here, use $\vec{C} = \vec{B}$ and $\vec{D} = -\vec{A}$:

$$\left|\left(\vec{B} - \vec{A}\right)\right| = \sqrt{(B_x - A_x)^2 + (B_y - A_y)^2}$$

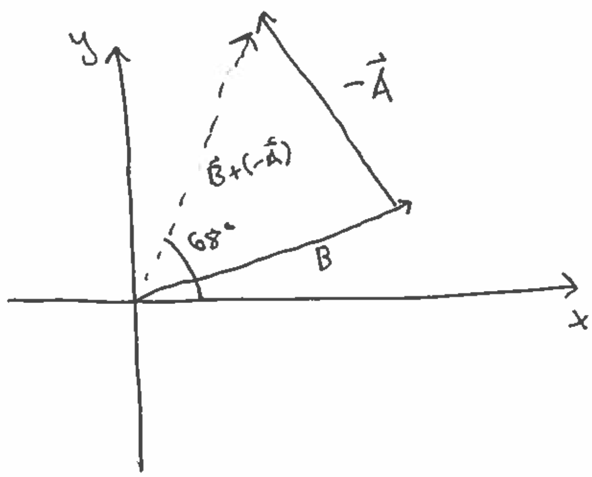
$$= \sqrt{(4)^2 + (10)^2}$$

$$|\vec{B} - \vec{A}| = 10.8$$

$$\theta_{\vec{B}-\vec{A}} = \arctan\left(\frac{10}{4}\right)$$

$$\theta_{\vec{B}-\vec{A}} = 68.2^\circ$$

Check by sketching:



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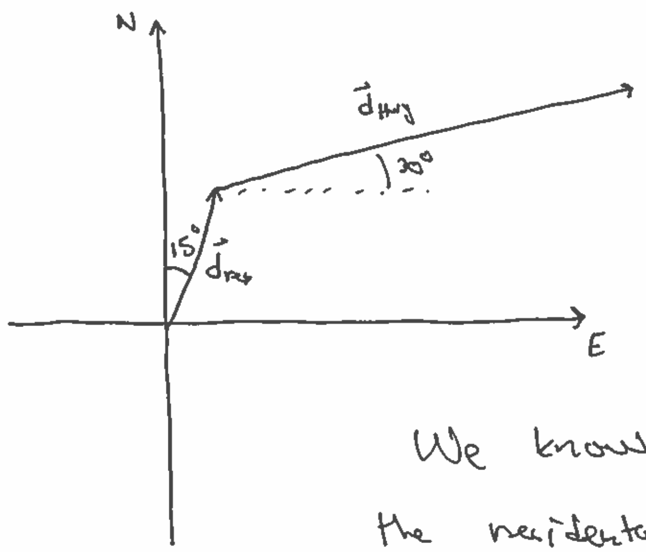
Problem 9: In general, Damien's displacement looks like this:



$\vec{d}_{res}$ = displacement along residential road

= ?? at 15° E of N

$\vec{d}_{hwy}$ = displacement along highway

= 15 km at 30° N of E.

We know that he spends $\Delta t_{res} = 15$ min on the residential road and $\Delta t_{total} = 24$ min total in transit.

(a) He spends $\Delta t_{hwy} = \Delta t_{tot} - \Delta t_{res} = 9$ min on the highway, so

$\vec{v}_{av} = \dfrac{\vec{d}_{hwy}}{\Delta t_{hwy}} = \dfrac{15 \text{ km}}{9 \text{ min}}$ at 30° N of E. We are to report this in [m/s], so

$$|\vec{v}_{av}| = \frac{15}{9} \frac{\text{km}}{\text{min}} \left(\frac{1 \text{ min}}{60 \text{ s}}\right)\left(\frac{1000 \text{ m}}{1 \text{ km}}\right) = 27.8 \text{ m/s}$$

$$\boxed{\vec{v}_{av} = 27.8 \text{ m/s at } 30° \text{ N of E}}$$

(b) Damien travels at 30 mph for 15 minutes (or $\frac{1}{4}$ hrs), so he travels a distance of

$$|\vec{d}_{res}| = 30 \frac{\text{mi}}{\text{hr}}\left(\frac{1}{4} \text{ hr}\right)$$

along the residential road.

$= 7.5$ miles.

Using 1 mile = 1.6 km.

$|\vec{d}_{res}| = 12$ km

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We may thus calculate the total distance by adding the magnitude of the two displacements:

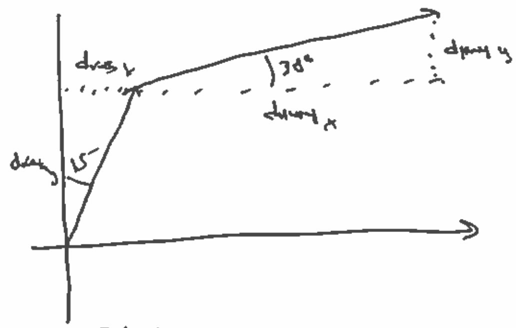
$$d_{tot} = |\vec{d}_{res}| + |\vec{d}_{hwy}|$$
$$= 12\text{ km} + 15\text{ km}$$
$$= 27\text{ km}$$

$$\boxed{d_{tot} = 27000\text{m}}$$

To get the total displacement, we must add the two displacements as vectors:

$$\vec{d}_{tot} = \vec{d}_{res} + \vec{d}_{hwy}$$

$$\begin{cases} d_{tot\,x} = d_{res,x} + d_{hwy,x} \\ d_{tot\,y} = d_{res,y} + d_{hwy,y} \end{cases}$$



We see that
$$d_{res,x} = d_{res}\sin 15° = 3106\text{ m}$$
$$d_{res,y} = d_{res}\cos 15° = 11591\text{ m}$$

$$d_{hwy,x} = d_{hwy}\cos 30° = 12990\text{ m}$$
$$d_{hwy,y} = d_{hwy}\sin 30° = 7500\text{ m}$$

$$\longrightarrow \quad \vec{d}_{tot} = (d_{res,x} + d_{hwy,x})\hat{x} + (d_{res,y} + d_{hwy,y})\hat{y}$$

$$\boxed{\vec{d}_{tot} = (16096\,\hat{x} + 19091\,\hat{y})\text{ m}}$$

or $\vec{d}_{tot} = 24971\text{m}$ at $50°$ N of E.

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(c) To get the speed, simply divide the total distance by Δt_{tot}:

$$\boxed{\text{Speed av.} = \frac{2700\,m}{24\,min\left(\frac{60\,s}{1\,min}\right)} = 18.75\ m/s.}$$

Average velocity is then $\frac{\vec{d}_{tot}}{\Delta t_{tot}}$, which is

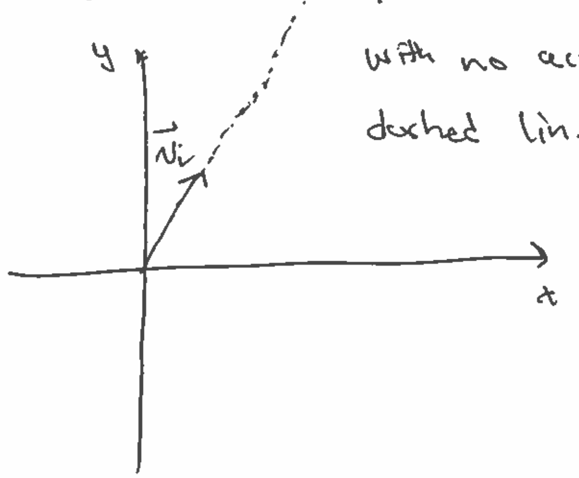
$$|\vec{v}_{av}| = \frac{24\,971\,m}{24\,min\left(\frac{60\,s}{1\,min}\right)} = 17.34\ m/s$$

$$\boxed{\vec{v}_{av} = 17.34\ m/s \text{ at } 50^\circ \text{ N of E}}$$

Problem 10: Given $\vec{v}_i = (3.0\hat{x} + 2.0\hat{y})$ m/s and

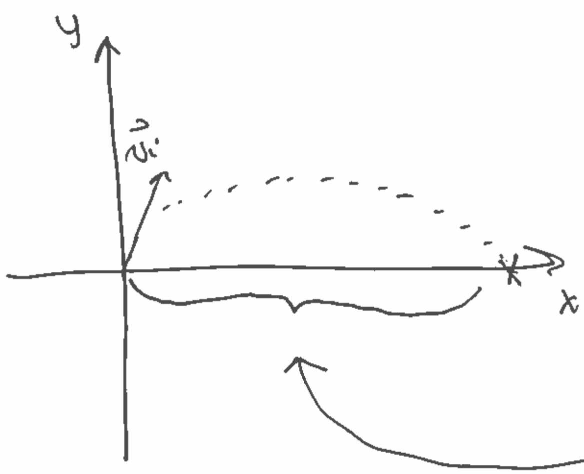
$\vec{a} = (5.0\hat{x} + a_y\hat{y})$ m/s^2,

We see that the particle begins traveling up and to the right:



With no accel., the object would travel along the dashed line. (and never cross the x-axis after $t=0$!)

For it to cross +x-axis, it must have a negative y-comp. to its acceleration! Then it would do something like this:



Thus $\boxed{a_y < 0}$

We also know that it crosses the +x-axis when it is 25 m away from the origin!

Also, being on the x-axis means that $\Delta y = 0$! This gives us enough info to solve:

We want a_y such that $\Delta x = 25$ m when $\Delta y = 0$!

As $\vec{a}$ is constant, we may use kinematics:

$$\Delta x = v_{ix}\Delta t + \frac{1}{2}a_x(\Delta t)^2$$

$$\Delta y = v_{iy}\Delta t + \frac{1}{2}a_y(\Delta t)^2$$

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We may use the eq. for Δx to get time:

$$\Delta x = 25\,m = (3.0\,m/s)\,\Delta t + \frac{1}{2}(5.0\,m/s^2)(\Delta t)^2$$

$$\longrightarrow \Delta t = \frac{(-3.0\,m/s) \pm \sqrt{9.0 - 4\left(\frac{1}{2}\times 5.0\right)(-25)}}{5.0}$$

$$= -3.82\,s \text{ or } 2.62\,s$$ we want $\Delta t > 0$.

$$\Delta t = 2.62\,s.$$

Now we may use this to find a_y, as we know that $\Delta y = 0$ at this time:

$$\Delta y = 0 = (2.0\,m/s)(2.62\,s) + \frac{1}{2} a_y (2.62\,s)^2$$

$$\longrightarrow a_y = -1.53\,m/s^2.$$ [check: $a_y < 0$!]

Now we know the full acceleration vector:

$$\vec{a} = [5\hat{x} - 1.53\hat{y}]\,m/s^2$$ we can calculate.

$$\boxed{\begin{aligned} |\vec{a}| &= \sqrt{5^2 + (-1.53)^2} = 5.23\,m/s^2 \\ \Theta_{\vec{a}} &= \arctan\left(\frac{-1.53}{5}\right) = -17.0^\circ \text{ or } 343^\circ \end{aligned}}$$