Lesson 10: Collisions, Rotational Kinetic Energy, Torque, Center of Gravity (Sections 7.8 – 8.3)

Last time we used Newton's second law to develop the impulse-momentum theorem. In words, the theorem states that the change in linear momentum is caused by the impulse. In an equation,

$$\Delta\vec{\mathbf{p}} = \vec{\mathbf{J}}_{total} = \sum \vec{\mathbf{F}}\Delta t$$

If no external forces act, the total linear momentum of the system is conserved. This occurs in collisions and explosions.

We derived the equations for a one dimensional elastic collision (kinetic energy is conserved)

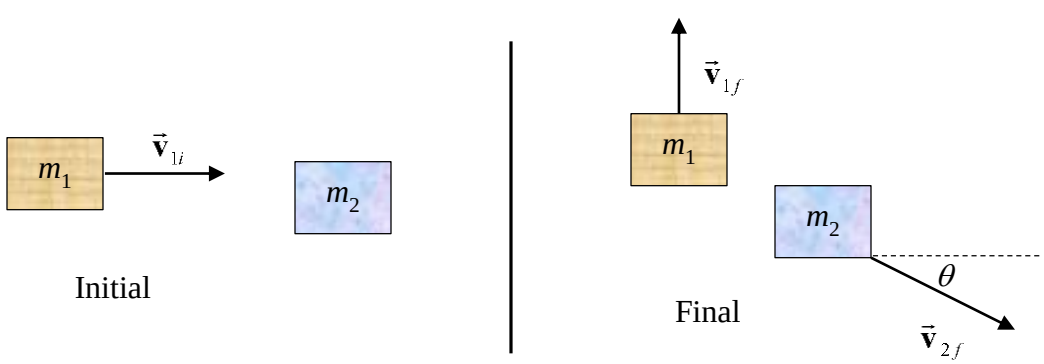
$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_{2i}$$

On many occasions the collision is not one dimensional. What do we do then?

Problem 82. An object of mass 2.0 kg (the "projectile") approaches a stationary object (the "target") at 8.0 m/s. The projectile is deflected through an angle of 90.0° and its speed after the collision is 6.0 m/s. What is the speed of the target body after the collision if the collision is perfectly elastic?

Solution: Since the incident mass changes direction after the collision, we cannot use the equations for an elastic one-dimensional collision. Instead, we have an elastic two-dimensional collision.



Since this is a collision, there are no external forces acting and momentum is conserved.

$$\vec{\mathbf{p}}_i = \vec{\mathbf{p}}_f$$

Using the diagram to take components:

$$p_{ix} = p_{fx} \qquad\qquad p_{iy} = p_{fy}$$

$$m_1 v_{1i} = m_2 v_{2f} \cos\theta \qquad\qquad 0 = m_1 v_{1f} - m_2 v_{2f} \sin\theta$$

$$m_1 v_{1f} = m_2 v_{2f} \sin\theta$$

The unknowns are θ, v_{2f} and m_2. There is a trick to eliminate θ. Square the two equations

$$(m_1 v_{1i})^2 = (m_2 v_{2f} \cos\theta)^2 \qquad\qquad (m_1 v_{1f})^2 = (m_2 v_{2f} \sin\theta)^2$$

$$m_1^2 v_{1i}^2 = m_2^2 v_{2f}^2 \cos^2\theta \qquad\qquad m_1^2 v_{1f}^2 = m_2^2 v_{2f}^2 \sin^2\theta$$

and add them together

$$\begin{aligned} m_1^2 v_{1i}^2 + m_1^2 v_{1f}^2 &= m_2^2 v_{2f}^2 \cos^2\theta + m_2^2 v_{2f}^2 \sin^2\theta \\ &= m_2^2 v_{2f}^2 \left(\cos^2\theta + \sin^2\theta\right) \\ &= m_2^2 v_{2f}^2 \end{aligned}$$

Above we used the identity

$$\cos^2\theta + \sin^2\theta = 1$$

Since the collision is elastic, kinetic energy is conserved:

$$\begin{aligned} K_i &= K_f \\ K_{1i} &= K_{1f} + K_{2f} \\ \tfrac{1}{2} m_1 v_{1i}^2 &= \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2 \\ m_1 v_{1i}^2 &= m_1 v_{1f}^2 + m_2 v_{2f}^2 \end{aligned}$$

Solving for m_2

$$\begin{aligned} m_1 v_{1i}^2 &= m_1 v_{1f}^2 + m_2 v_{2f}^2 \\ m_1 v_{1i}^2 - m_1 v_{1f}^2 &= m_2 v_{2f}^2 \\ m_2 &= \frac{m_1\left(v_{1i}^2 - v_{1f}^2\right)}{v_{2f}^2} \end{aligned}$$

and substitute into the equation found when we squared and added the momentum equations

$$m_1^2 v_{1i}^2 + m_1^2 v_{1f}^2 = m_2^2 v_{2f}^2$$

$$m_1^2 v_{1i}^2 + m_1^2 v_{1f}^2 = \left[\frac{m_1\left(v_{1i}^2 - v_{1f}^2\right)}{v_{2f}^2}\right]^2 v_{2f}^2$$

$$m_1^2\left(v_{1i}^2 + v_{1f}^2\right) = \frac{m_1^2\left(v_{1i}^2 - v_{1f}^2\right)^2}{v_{2f}^4} v_{2f}^2$$

$$v_{1i}^2 + v_{1f}^2 = \frac{\left(v_{1i}^2 - v_{1f}^2\right)^2}{v_{2f}^2}$$

$$v_{2f}^2 = \frac{\left(v_{1i}^2 - v_{1f}^2\right)^2}{v_{1i}^2 + v_{1f}^2}$$

$$v_{2f} = \sqrt{\frac{\left(v_{1i}^2 - v_{1f}^2\right)^2}{v_{1i}^2 + v_{1f}^2}} = \sqrt{\frac{\left((8.0\,\text{m/s})^2 - (6.0\,\text{m/s})^2\right)^2}{(8.0\,\text{m/s})^2 + (6.0\,\text{m/s})^2}} = 2.80\,\text{m/s}$$

Example: Consider a two dimensional elastic collision between two identical masses. One mass is initially at rest.

Solution: Elastic collisions conserve kinetic energy

$$\tfrac{1}{2} m_1 v_{1i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2$$

Cancelling the ½ and the masses, since $m_1 = m_2$,

$$\boxed{v_{1i}^2 = v_{1f}^2 + v_{2f}^2}$$

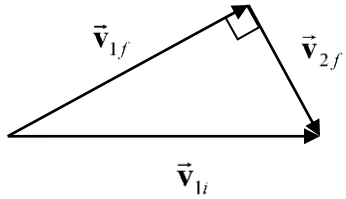
The conservation of linear momentum condition is

$$m_1 \vec{\mathbf{v}}_{1i} = m_1 \vec{\mathbf{v}}_{1f} + m_2 \vec{\mathbf{v}}_{2f}$$

Again, we can cancel the masses

$$\boxed{\vec{\mathbf{v}}_{1i} = \vec{\mathbf{v}}_{1f} + \vec{\mathbf{v}}_{2f}}$$

The two boxed equations can be interpreted by the picture



After the collision the two masses have velocities perpendicular to each other.

Chapter 8 Torque and Angular Momentum

Review of Chapter 5
We had a table comparing parameters from linear and rotational motion. Today we fill in the table. Here it is

Description	Linear	Rotational
position	x	θ
displacement	Δx	$\Delta\theta$
Rate of change of position	v_x	ω
Average rate of change of position	$v_{x,av} = \frac{\Delta x}{\Delta t}$	$\omega_{av} = \frac{\Delta\theta}{\Delta t}$
Instantaneous rate of change of position	$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$	$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$
Average rate of change of speed	$a_{x,av} = \frac{\Delta v_x}{\Delta t}$	$\alpha_{av} = \frac{\Delta\omega}{\Delta t}$
Instantaneous rate of change of speed	$a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t}$	$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$
Inertia	m	I
Influence that causes acceleration	F	τ
Momentum	$\vec{\mathbf{p}}$	$\vec{\mathbf{L}}$

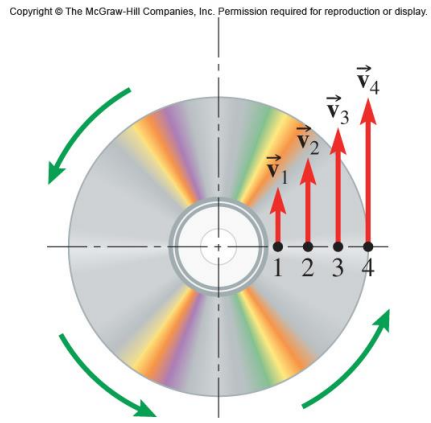
Lesson 10: Collisions, Rotational Kinetic Energy, Torque, Center of Gravity (Sections 7.8 – 8.3)

The relations (often physical laws) for rotational motion can be found by a simple substitution of rotational variables for the corresponding linear variables.

Rotational Kinetic energy

A wheel suspended at its axis can spin in space. Since the points of the wheel are moving, the wheel has kinetic energy.

All the pieces in a **rigid body** remain at the same location relative to all the other pieces. For a rotating object, the parts further away from the axis of rotation are moving faster.



$$v = r\omega$$

The total kinetic energy of all the pieces will be

$$K_{total} = \tfrac{1}{2} m_1 v_1^{\,2} + \tfrac{1}{2} m_2 v_2^{\,2} + \cdots + \tfrac{1}{2} m_N v_N^{\,2}$$

$$= \sum_{i=1}^{N} \tfrac{1}{2} m_i v_i^{\,2}$$

$$K_{total} = \sum_{i=1}^{N} \tfrac{1}{2} m_i r_i^{\,2} \omega^2$$

$$= \tfrac{1}{2}\left(\sum_{i=1}^{N} m_i r_i^{\,2}\right)\omega^2$$

The quantity in parentheses is called the **rotational inertia** (or the moment of inertia)

$$I = \sum_{i=1}^{N} m_i r_i^{\,2}$$

Finding the Rotational Inertia (page 270)

1. If the object consists of a *small* number of particles, calculate the sum

$$I = \sum_{n=1}^{N} m_n r_n^{\,2}$$

   directly.
2. For symmetrical objects with simple geometric shapes, calculus can be used to perform the sum. Table 8.1 (see below) lists the results of these calculations for the shapes most commonly encountered.
3. Since the rotational inertia is a sum, you can always mentally decompose the object into several parts, find the rotational inertia of each part, and then add them. This is an example of the *divide-and-conquer* problem-solving technique.

The rotational inertia depends on the location of the rotation axis. The same object will have a different rotational inertia depending on where it is rotating. Look at the formula for a thin rod below.

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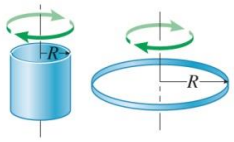
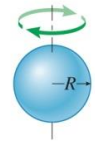
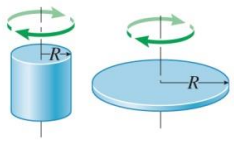
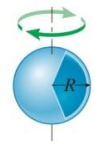
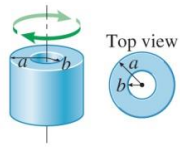
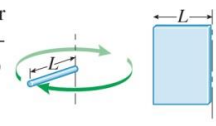
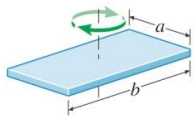
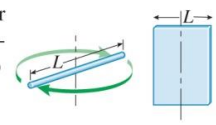
Table 8.1 Rotational Inertia for Uniform Objects with Various Geometrical Shapes

Shape		Axis of Rotation	Rotational Inertia	Shape		Axis of Rotation	Rotational Inertia
Thin hollow cylindrical shell (or hoop)		Central axis of cylinder	MR^2	Solid sphere		Through center	$\frac{2}{5}MR^2$
Solid cylinder (or disk)		Central axis of cylinder	$\frac{1}{2}MR^2$	Thin hollow spherical shell		Through center	$\frac{2}{3}MR^2$
Hollow cylindrical shell or disk		Central axis of cylinder	$\frac{1}{2}M(a^2+b^2)$	Thin rod (or rectangular plate)		Perpendicular to rod through end (or along edge of plate)	$\frac{1}{3}ML^2$
Rectangular plate		Perpendicular to plate through center	$\frac{1}{12}M(a^2+b^2)$	Thin rod (or rectangular plate)		Perpendicular to rod through center (or parallel to edge of plate through center)	$\frac{1}{12}ML^2$

The rotational kinetic energy of a rigid object rotating with angular velocity ω is

$$K = \tfrac{1}{2} I\omega^2$$

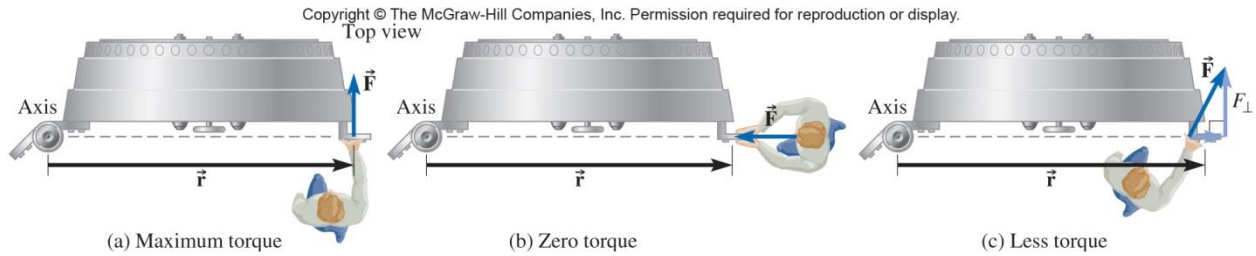
Compare to the translational kinetic energy

$$K = \tfrac{1}{2} mv^2$$

Torque

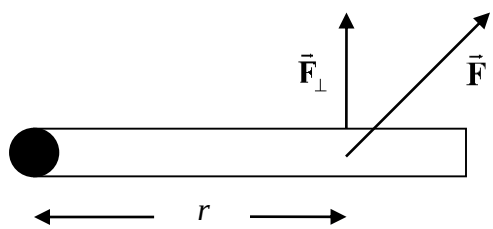
A quantity related to force, called torque, plays the role in rotation that force itself plays in translation. A torque is not separate from a force; it is impossible to exert a torque without exerting a force. Torque is a measure of how effective a given force is at twisting or turning something.

The torque due to a force depends of the magnitude of the applied force, the force's point of application, and the force's direction.



First definition of torque

$$\tau = \pm r F_\perp$$

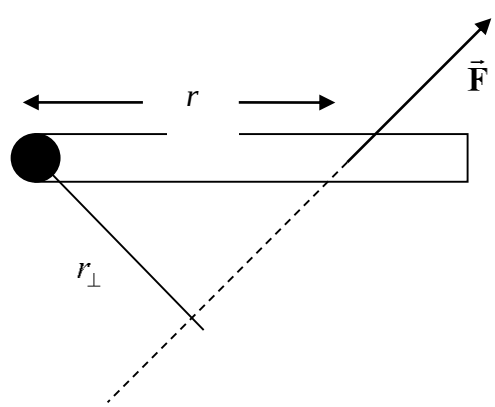


Because rotations have directions, we assign the + sign to torques that cause counterclockwise rotations, and – sign to torques that cause clockwise rotations. What is the sign of the torque in the figures above?

Torques are measured in the units of force times distance. This is the same dimensions as work. However, torque has a different effect than work. To keep the two concepts distinct, we measure work in joules and torque in newton-meters.

Second definition of torque

$$\tau = \pm r_\perp F$$



Find the **lever arm** (or **moment arm**) by extending the line of the force and drawing a line from the axis of rotation so that is crosses the line of the force at a right angle. Finding the lever arm is often <u>the most difficult part</u> of a torque problem.

Finding Torques Using the Lever Arm (p 278)

1. Draw a line parallel to the force through the force's point of application; this line is called the force's **line of action**.
2. Draw a line from the rotation axis to the line of action. This line must be perpendicular to both the axis and the line of action. The distance from the axis to the line of action along this perpendicular line is the lever arm ($r_\perp$). If the line of action of the force goes through the rotation axis, the lever arm and the torque are both zero.
3. The magnitude of the torque is the magnitude of the force times the lever arm:

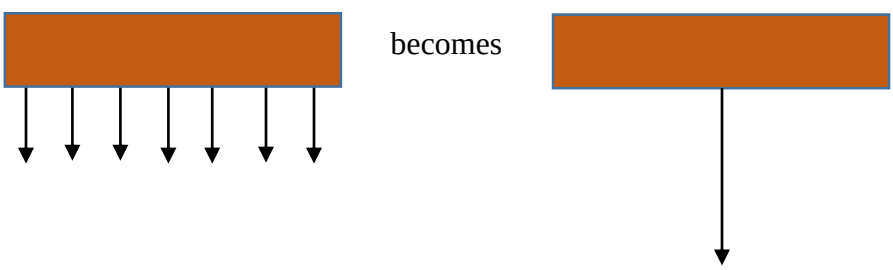
$$\tau = \pm r_\perp F$$

4. Determine the algebraic sign of the torque as before.

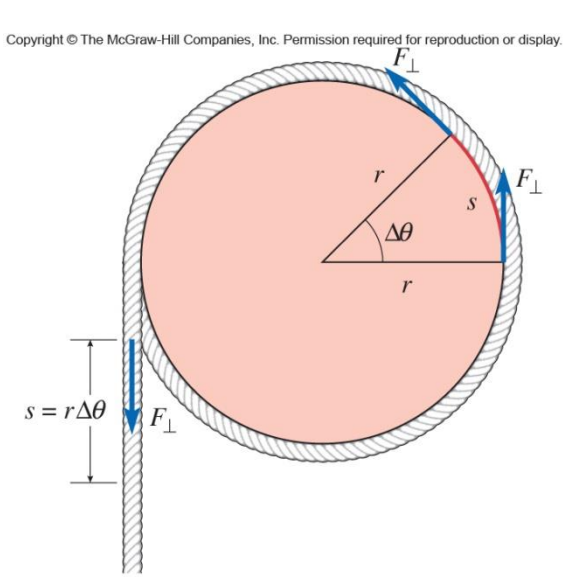
Center of Gravity

When the gravitational force acts on an object, all the small pieces of the object experience the gravitational force. This very large number of forces taken around an axis will create a torque. How do we deal with that?

Fortunately, we can greatly simplify the problem. The total force can be considered to act a single point called the **center of gravity**. If the gravitational field is uniform in magnitude and direction, the center of gravity is located at the center of mass.



Work done by a torque



The expression for the work done is

$$\begin{aligned}\Delta W &= F_{\perp}\Delta s \\ &= \left(\frac{\tau}{r}\right)\left(r\Delta\theta\right) \\ &= \tau\Delta\theta\end{aligned}$$

Power is the rate of doing work

$$\begin{aligned}P &= \frac{\Delta W}{\Delta t} \\ &= \frac{\tau\Delta\theta}{\Delta t} \\ &= \tau\omega\end{aligned}$$