L02 Dimensional Analysis and Guesstimation

Physical Constants You Should Memorize.

$$\hbar \approx 1 \times 10^{-34} \text{ Js } [ML^2T^{-1}]$$
$$c \approx 3 \times 10^8 \text{ ms}^{-1} [LT^{-1}]$$
$$e \approx 1.6 \times 10^{-19} \text{ C } [IT]$$
$$m_e \approx 9 \times 10^{31} \approx 10^{-30} \text{ kg } [M]$$
$$\mu_o = 4\pi \times 10^{-7} \text{ Hm}^{-1} [??]$$
$$\epsilon_o = 1/(\mu_o c^2) \text{ Fm}^{-1} [??]$$
$$k_B \approx 1.4 \times 10^{-23} \text{ JK}^{-1} [ML^2T^{-2}\theta^{-1}]$$
$$g = 9.8 \text{ ms}^{-2} [LT^{-2}] \text{ (directly related to } G)$$

1. *Dimensional analysis* offers a method for reducing complex physical problems to the simplest form w/o going through rigorous modeling or calculations. This is an important process in understanding the essence of a physical phenomenon and/or designing an experiment for an investigation. Sometimes the results from dimensional analysis could be the only ones you can possibly get.

Bridgman explains it: *The principle use of dimensional analysis is to deduce from a study of dimensions of variables in any physical system certain limitations on the form of any possible relationship between these variables.*

A wonderful example of DA is Taylor's analysis of the atomic bomb tested in Trinity in the clandestine Manhattan project (PPT slides).

2. Physical Quantities and Dimension

Base Quantity and Derived Quantity
Science including physics begins with observations of phenomena. Through rigorous and controlled experimentation and logical thought process, the physical phenomena are described quantitatively using mathematical tools. Any quantitative description of a property requires comparison with a reference, for example length needs a meter-stick. In this process we recognize a very obvious fact that properties of different kinds cannot be compared.

The allowed types of physical properties are called *Physical Quantities*. There are two classes of physical quantities: 1. Basic Quantities and 2. Derived Quantities. The physical base quantity of a certain kind **A** can be represented by the multiplication of a number A and the unit quantity of the kind $\hat{a}$,
$$\mathbf{A} = A\hat{a}$$

Given the type of base quantity, the choice of the unit is arbitrary; mass m can be in gram, kilogram, or pound. Obviously the the numerical values of the quantity varies depending on the choice of the unit. However, the quantity itself does not change.

$$\mathbf{A} = A\hat{a} = A'\hat{a}'.$$

Here, we know if $\hat{a}' = n\hat{a}$, then $A' = n^{-1}A$. Therefore, it is crucial to know which unit of the physical quantity is used.

There are only limited number of base quantities such as time, mass, length, temperature, angle, and so on. From these base quantities, one can also generate in principle infinite number of physical quantities called derived quantities. Let us say that we measured the distance of travel $\mathbf{L}$ of a vehicle and the time took $\mathbf{t}$. From the measurements, we can calculate a new quantity, the average speed of the travel, $\mathbf{v} = \mathbf{L}/\mathbf{t}$. So a derived quantity *speed* is defined. We can also calculate another derived (physical) quantity, kinetic energy E_k using another base quantity, mass M and the derived quantity speed:

$$\mathbf{E_k} = \frac{1}{2}\mathbf{m}\mathbf{v}^2.$$

Here I use bold face variables to emphasize that the quantities include proper units or unit independent. Below I make a very important statement without providing the proof.

All monomial derived physical quantities have the following power-law form.

$$Q = \alpha A^a B^b C^c \cdots,$$

where Q is the derived quantity and A, B, C, ... are base quantities. α is a numerical constant and a, b, c ... are real numbers.

Dimension

Each type of base physical quantity has its own dimension. If A is the numerical value of a length, we say it has the dimension of *Length*:

$$[A] = L.$$

Dimension of a derived quantity $Q = \alpha L_1^{\ell_1} L_2^{\ell_2} \cdots M_1^{m_1} M_2^{m_2} \cdots T^{\tau_1} T^{\tau_2} \cdots$ is given by

$$[Q] = (L)^{\Sigma \ell_i}(M)^{\Sigma m_i}(T)^{\Sigma \tau_i} \cdots$$

e. g. Dimension of kinetic energy $[E_k] = ML^2T^{-2}$. And all kinds of energy should have exactly the same dimension.

1. Any derived quantities involving special functions such as ln *,* sin *, exp, ... should have their argument* **dimensionless**.

2. Physical equations must be **dimensionally homogeneous**: *both sides of an equation should have the same dimension.*

3. System of Units

What are the fundamental base quantities that one can describe physical phenomena properly? The system of units is composed of a complete set of base quantities, the base

units, and all relevant derived quantities–of course they are open ended! There could be many different sets of system of units but Systeme International (SI) system is a unit system internationally adopted.

TABLE I: Dimensions of some commonly used physical quantities in the $LMT\theta I$-class

Property	Symbol	B or D	Dimension
Length	L	B	L
Mass	m	B	M
Time	t	B	T
Temperature	T	B	θ
Current	I	B	I
Number	N	B	1
Force	F	D	MLT^{-2}
Pressure	P	D	$ML^{-1}T^{-2}$
Energy	E	D	ML^2T^{-2}
Electric Charge	Q	D	IT
Electric Field	E	D	$MLT^{-3}I^{-1}$
Magnetic Field	B	D	$MT^{-2}I^{-1}$
Angle	α	B	1

The SI system is composed of
6 base quantities:
and derived quantities:

4. How and where to use DA? *Check if the equation is correct.*

Check if your answer is correct and physically meaningful.

Extract expression without rigorous calculations.

5. Buckingham's Π-Theorem (Optional)

The above examples give you an idea how to use DA in expressing a physical quantities in terms of other physical quantities. These are the simplest kind of examples. There is a general principle that you can extract the dependencies in a systematic way: Buckingham's Π Theorem. At the heart of DA is the *dimensional homogeneity* of any physically significant equations relating various physical properties.

Let's consider a dependent physical quantity Q_o–this quantity depends on other physical quantities. Now the task is how to extract the specific dependence of each physical quantity in Q_o.

Step 1: Identify a complete set of independent quantities $\{Q_1, Q_2, ..., Q_n\}$ that determines Q_o:

$$Q_o = f(Q_1, Q_2, ...).$$

$\{Independent \quad Variables\}$: the value of each member can be adjusted arbitrarily without affecting the values of others in the set.

For example, a set of variables describing the motion of an object E_k, v, m is not a set of independent variables because if you change the speed v, the kinetic energy of the object E_k also changes. One would like to extract the period T of a pendulum (a mass on a string). My physical intuition says $T = f(\ell, g)$. In this case $\{\ell.g\}$ is an independent variable set!

Step 2: Dimensional Consideration (Homogeneity)

One should know the dimension of all variables Q_o, Q_1, Q_2, ..., Q_n:

$$[Q_i] = L^{\ell_i} M^{m_i} t^{\tau_i}.$$

Then, identify a subset of *dimensionally independent* variables from the set of independent variables:

$$\{Q_1, Q_2, ..., Q_n\} \longrightarrow \{Q_1, Q_2, ..., Q_k\} \quad \{Q_{k+1}, ..., Q_n\}.$$

Here, the dimension of $Q_{k+1}, ..., Q_n$ can be expressed by a combination of $\{Q_1, Q_2, ..., Q_k\}$. There two possible cases, $k = n$ or $k < n$. If $k = n$, then all the independent variables are also dimensionally independent and the following is straightforward. If not, then we have to one more step. Basically the idea is the dimensionally independent variables should determine $[Q_o]$ and the other dimensionally dependent variables should appear in the form of dimensionless variables.

EX: Is $\{\rho, v, f\}$ a physically independent set? Is it a dimensionally independent set?

EX: m, E_k, v are not physically independent not dimensionally independent.

EX: Let's revisit the pendulum case. $T = f(\ell, m)$. It is clear here that the chosen variables are physically and dimensionally independent. Then the homogeneity dictates

$$[T] = [\ell]^x g^y = L^x L^y T^{-2y}.$$

Since $[T] = T^{-1}$, we solve

If $k < n$, find the combination of dimensionally independent variables for each dimensionally dependent variable following the similar process:

$$[Q_{k+1}] = [Q_1^{N_1} Q_2^{N_2} \cdots Q_k^{N_k}],$$

and so on. Then define a new dimensionless variable that would replace the dimensionally independent variable:

$$\Pi_{k+1} =$$

$Q_1^{N_1} Q_2^{N_2} \cdots Q_k^{N_k}.$

Finally,

$$Q_o = Q_1^{\alpha} Q_2^{\beta} \cdots Q_k^{\kappa} \Phi(\Pi_{k+1}, \cdots, \Pi_n),$$

where the function Φ is unknown at this point but can be determined through rigorous calculations. Note that Φ is dimensionless!

EX: In the pendulum example, one project that the period of the pendulum should depend on m, g, and ℓ. If you do DA with $T = h(m, g, \ell)$, where do you reach?

EX: When an object moves in air at a relatively high velocity, it will feel a drag force, f_d. One can choose that the drag force is related to the density of air ρ, the size of the object (diameter for a sphere) D, the speed of the object v, and also the sound speed of air c. The sound of speed may not be so obvious but let's take it for now.

(1) Are they physically independent? *Yes, but not dimesionally independent*

(2) Choose a dimensionally independent set. $\{\rho, D, v\}$

(3) Do DA and find $f_d = \rho^x D^y v^z \Phi(\Pi_{[}1)$.