Second order homogeneous equations with constant coefficients

\[ \ddot{u} + b \dot{u} + cu = 0 \quad \rightarrow \quad (1) \]

homogeneous means right hand side of (1) is zero. You can use whichever method you feel comfortable with.

Method 1: let \( u = e^{\lambda t} \)
\[ \Rightarrow \quad \ddot{u} = \lambda^2 e^{\lambda t} \quad , \quad \dot{u} = \lambda e^{\lambda t} \]
Replacing in (1) we get
\[ \lambda^2 e^{\lambda t} + b \lambda e^{\lambda t} + c e^{\lambda t} = 0 \]
\[ \Rightarrow \quad \lambda^2 + b \lambda + c = 0 \quad \rightarrow \quad (2) \]
\[ \lambda_1 = \frac{1}{2} \left( -b + \sqrt{b^2 - 4c} \right) \quad , \quad \lambda_2 = \frac{1}{2} \left( -b - \sqrt{b^2 - 4c} \right) \quad \rightarrow \quad (3) \]

\[ \Rightarrow \quad \text{Second order homogeneous eqn. always has two independent solutions, which are} \]
\[ u = e^{\lambda_1 t} \quad \text{&} \quad e^{\lambda_2 t} \quad \text{and the general solution is:} \]
\[ u = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \]

Method 2: let \( D = \frac{d}{dt} \) (D is an operator like \( \nabla \)).

\[ \Rightarrow \quad \text{Eqn. 1 can be written as:} \]
\[ D^2 u + bD u + c u = 0 \]
\[ \Rightarrow \quad (D^2 + bD + c) u = 0 \quad , \quad \text{The operator} \quad D^2 + bD + c \quad \text{can be factored using the roots of equation (2) or you can find the roots independently.} \]

\[ \Rightarrow \quad (D - \lambda_1)(D - \lambda_2) u = 0 \quad \Rightarrow \quad \text{Either} \quad (D - \lambda_1) u = 0 \quad \rightarrow \quad (3) \]
\[ \text{or} \quad (D - \lambda_2) u = 0 \quad \rightarrow \quad (4) \]
from (3), \( (D - \lambda_1) u = 0 \) \[ \Rightarrow \quad \frac{du}{dt} - \lambda_1 u = 0 \]
\[ \Rightarrow \quad \frac{du}{dt} = \lambda_1 t \quad \Rightarrow \quad \ln u = \lambda_1 t \quad \Rightarrow \quad u = e^{\lambda_1 t} \]
and from (4) \( u = e^{\lambda_2 t} \),

I will use this method.
If \( \lambda_1 \) and \( \lambda_2 \) are solutions independent solutions for \( x \) then the general solution for \( x \):

\[ x = A \lambda_1 + B \lambda_2 \]

the constants are determined from the initial conditions.

**Case I** if \( B^2 > 4c \), then \( \lambda_1 \) and \( \lambda_2 \) are real.

**Case II** if \( B^2 < 4c \), then \( \lambda_1 \) and \( \lambda_2 \) are complex.

This is the case for simple harmonic oscillations.

then \( x = A e^{\lambda_1 t} + B e^{\lambda_2 t} \) just like case I.

**Case III** \( B^2 = 4c \), then \( \lambda_1 = \lambda_2 \) from (2).

\[ \Rightarrow \lambda_1 = \lambda_2 = -\frac{b}{2c} \]

\[ \Rightarrow x = e^{-\frac{b}{2c} t} \]

But this is just one solution for (1) but its got to have 2 solutions because its a second order diff. eq.

To find the 2nd solution:

let \( x_2 = u x_1 \) where \( u \) is a function of \( \beta t \).

Substituting in (1):

\[ \ddot{x}_2 + bx_2 + c = 0 \]

\[ \Rightarrow \ddot{u} x_1 + 2 \dot{u} \dot{x}_1 + 2u \ddot{x}_1 + b \dot{u} x_1 + b u \dot{x}_1 + c u x_1 = 0 \]

\[ \Rightarrow \ddot{u} (x_1) + \ddot{u} (2x_1 + bx_1) + u (\dddot{x}_1 + b \dot{x}_1 + cx_1) = 0 \Rightarrow (5) \]

\[ \dddot{x}_1 + b \dot{x}_1 + cx_1 = 0 \] because \( x_1 \) is a solution of (1).

Also \( 2x_1 = \frac{d}{dt} (e^{-\frac{b}{2c} t}) = -be^{-\frac{b}{2c} t} = -bx_1 \)

\[ 2 \dot{x}_1 + bx_1 = 0 \]

\[ \Rightarrow \dot{u} x_1 = 0 \Rightarrow \ddot{u} = 0 \Rightarrow u = c_1 t + c_2. \]

So for the second solution we can take \( u = t \)

\[ x_2 = t e^{-\frac{b}{2c} t}. \] You can check that this is a solution of (1).
Non-homogeneous eqn.:
\[ \ddot{x} + 5\dot{x} + 6x = f(t). \quad \rightarrow \quad (6) \]
The right hand side is not equal to zero. Then the general solution is given by:

\[ x = x_h + x_p, \quad \text{where } x_h \text{ is the general solution of the equation } \ddot{x} + 5\dot{x} + 6x = 0. \]
Replacing in (6):

\[ \ddot{x}_h + 5\dot{x}_h + 6x_h = f(t). \]

\[ \Rightarrow \ddot{x}_h + 5\dot{x}_h + 6x_h + 5\dot{x}_p + 6x_p = f(t). \]

This term is zero.

\[ \Rightarrow \ddot{x}_p + 5\dot{x}_p + 6x_p = f(t). \]

If \( f(t) \) is a constant (not a function of \( t \)), then

\[ f(t) = k, \quad \text{and } x_p = \frac{k}{c}, \quad \text{because } \left( \frac{k}{c} \right) + 5\left( \frac{k}{c} \right) + 6 \frac{k}{c} = 0 + 0 + k = k. \]

I will do other cases in class.

Practice:

\[ \ddot{x} - \dot{x} - 2x = 0 \quad \Rightarrow \quad (D^2 - D - 2)x = 0 \]

\[ \Rightarrow \quad (D - 2)(D + 1)x = 0 \]

\[ \Rightarrow \quad \lambda_1 = 2, \quad \lambda_2 = -1 \]

\[ \Rightarrow \quad \begin{align*}
x_1 &= e^{2t}, \\
x_2 &= e^{-t} \\
\end{align*} \]

\[ \Rightarrow \quad x = A e^{2t} + B e^{-t} \]
If \( \ddot{x} - \dot{x} - 2x = 10 \),

then the general solution is:

\[ x = x_h + x_p \]

where \( x_h \) is the solution for the equation:

\[ \ddot{x} - \dot{x} - 2x = 0 \], which we found earlier.

\( x_p \) is the particular solution of

\[ \ddot{x} - \dot{x} - 2x = 10 \]

Since the right hand side of the equation is a constant, it is easy to find \( x_p \).

Just let \( x_p = 5 \), then.

\[ \ddot{x}_p - \dot{x}_p - 2x_p = 0 - 0 + 2 \times 5 = 10 \]

So the general solution for \( x \) is:

\[ x = Ae^{2t} + Be^{-t} - 5 \]