

$$\bar{\alpha} = 0.34 \times 10^6 \frac{\text{m}^2}{\text{s}}$$

Problem:

Suppose that you are installing a water pipe in a location where the air temperature in the winter can be -5°C for time periods of up to 2 months. Find the min depth you must lay your pipe to ensure that it does not freeze, assuming the average soil temperature is initially 20°C .
 (-5°C can be assumed as constant and = to surf temp.)

Solution:

Use eqn. $\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$

- Problems: This eqn is difficult to solve, as is, analytically, since it is a PDE. One common method to solve it is the method of similarity solutions.

• This method is useful in this case because the system can be assumed semi-infinite and because we are able to combine the Initial condition with the B.C at ∞ to a single boundary condition.

Application of Buckingham's π Theorem:

Want: to use B. π . T to determine the minimum number of dimensionless variables needed to solve this problem which in turn implies that the min number of parameters are used.

start $\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$ [1]

eqn [1] can be written as

$$\rho c_p \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = 0$$

$$\Rightarrow f(\quad) = 0$$

I.C.: $t=0, T=T_i$

B.C.: $x=0, T=T_{\text{air}}$

$x=\infty, T=T_i$

• From our initial equation, I.C., and B.C.'s we find

$$f(\quad) = f(T, T_i, T_{\text{air}}, \rho, c_p, k, t, x) = 0$$

differential

Since this is a heat transfer problem temperature is expressed as a difference and density, heat capacity, and thermal conductivity are combined to appear as $\alpha = \frac{k}{\rho c_p}$, α is known as the thermal diffusivity, which is analogous to the kinematic viscosity of fluids, and mass diffusivity.

$$\Rightarrow f(T, T_i, T_{air}, \rho, c_p, k, x, t) = f(T - T_{air}, T_{air} - T_i, \alpha, x, t) = 0 \quad [2]$$

Now our function consists of 5 physical quantities and 3 fundamental units (length, time, temperature). From B. P. T. there are at most 2 dimensionless variables needed for this problem. To find these variables we will use B. P. T.

$$f(T - T_{air}, T_{air} - T_i, \alpha, x, t) = 0$$

can be re-written as

$$T - T_i = g(T_{air} - T_i, \alpha, x, t)$$

$$\Rightarrow [T - T_i] = [T_{air} - T_i]^a [\alpha]^b [x]^c [t]^d$$

using the Dorsey's relation

$$\phi = [\phi]^a [L^2 T^{-1}]^b [L]^c [T]^d$$

Doing a balance on the 4 fundamental units

$$\text{Temp } (\phi): \quad 1 = a - c \quad (a)$$

$$\text{time } (T): \quad 0 = c - b \quad (c)$$

$$\text{length } (L): \quad 0 = 2b + d \quad (d)$$

• We now have 3 equations and 4 unknowns.

From (a), $a = 1$

From (c), $c = b$

and from (d) $d = -2b = -2c$

Choosing d as the independent power we get,

$$[T - T_i] = [T_{air} - T_i]^{1 - \frac{1}{2}d} [\alpha]^{1 - \frac{1}{2}d} [x]^{1 - \frac{1}{2}d} [t]^d \quad \text{herefore}$$

$$\frac{[T - T_i]}{[T_{air} - T_i]} = \left[\frac{x}{\sqrt{4\alpha t}} \right]^2$$

Now $\frac{[T - T_i]}{[T_{air} - T_i]} = g\left(\frac{x}{\sqrt{4\alpha t}}\right)$

The choices for our dimensionless variables are
now

$$\theta \equiv \frac{[T - T_i]}{[T_{air} - T_i]} \quad \text{and} \quad z \equiv \frac{x}{\sqrt{4\alpha t}}$$

Note: The factor of $\sqrt{4}$ in the denominator in z is included because this is a well studied problem and it leads to a neat solution of the original equation. Without prior knowledge of the solution we would not have used this factor.

Solving the problem:

Now our original eqn. $\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$ or $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$
By nondimensionalization becomes:

$$\frac{\partial \theta}{\partial t} = -\frac{1}{2} z \frac{d\theta}{dz}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{4\alpha t} \frac{d^2 \theta}{dz^2}$$

substituting these back into our equation we get

$$[Eq. 3] \quad \frac{d^2 \theta}{dz^2} + 2z \frac{d\theta}{dz} = 0$$

Now the corresponding boundary and initial conditions become

$$z = 0, \quad \theta = 1$$

$$z = \infty, \quad \theta = 0$$

One final substitution of $\xi = \frac{d\theta}{dz}$ reduces [3] into a 1st order separable equation.

Upon integration we get: $\theta = c_1 \exp(-z^2) = \frac{d\theta}{dz}$
a second integration gives:

$$\theta = c_1 \int_0^z \exp(-z^2) dz + c_2$$

After applying our boundary conditions $c_2 = 1$ and
and

$$c_1 = - \frac{1}{\int_0^{\infty} \exp(-z^2) dz} = - \frac{2}{\sqrt{\pi}}$$

$$\Rightarrow \theta = 1 - \frac{2}{\sqrt{\pi}} \int_0^z \exp(-z^2) dz = 1 - \operatorname{erf}(z) = \operatorname{erfc}(z)$$

where erfc is the complementary error function or the gaussian distribution curve.

We can now use this relationship between temperature, depth, and time to solve our problem

We know $T_i = 20^\circ\text{C}$, $T_{\text{air}} = -5^\circ\text{C}$, and $T = 1^\circ\text{C}$

$$\Rightarrow \theta = \frac{T - T_i}{T_{\text{air}} - T_i} = \frac{-19^\circ\text{C}}{-25^\circ\text{C}} = 0.76$$

$$\therefore \text{from graph } z = 0.2 = \frac{x}{\sqrt{4\alpha t}}$$

$$t = 2 \text{ month} = 5.259488 \times 10^6 \text{ s}$$

$$\alpha = 0.34 \times 10^{-6} \frac{\text{m}^2}{\text{s}}$$

$$\Rightarrow x = 0.2 \sqrt{4(0.34 \times 10^{-6} \frac{\text{m}^2}{\text{s}})(5.259488 \times 10^6 \text{ s})}$$

$$= 0.535 \text{ m}$$

$$\boxed{x \leq 1.75 \text{ ft}}$$

This solution can analogously be applied to momentum or diffusion

Temp-Distribution in a semi-infinite slab

