$\bar{\alpha} = 0.34 \times 10^{-6} \frac{m^2}{s}$

Problem:

Suppose that you are installing a water pipe in a location where the air temperature in the winter can be $-5°C$ for time periods of up to 2 months. Find the min depth you must lay your pipe to ensure that it does not freeze, assuming the average soil temperature is initially $20°C$.

($-5°C$ can be assumed as constant and = to surf temp.)

Solution:

Use eqn. $\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$

- Problems: this eqn is difficult to solve, as is, analytically, since it is a PDE. One common method to solve it is the method of similarity solutions.
  - This method is useful in this case because the system can be assumed semi-infinite and because we are able to combine the Initial condition with the B.C at ∞ to a single boundry condition.

Application of Buckingham's π Theorem:

Want: to use B.π.T to determine the minimum number of dimensionless variables needed to solve this problem which inturn implies that the min number of parameters are used.

Start

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \quad [1]$$

I.C.: $t = 0$, $T = T_i$

B.C.: $x = 0$, $T = T_{air}$

$x = \infty$, $T = T_i$

eqn [1] can be written as

$$\rho C_p \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = 0$$

$$\Rightarrow f(\qquad) = 0$$

- from our initial equation, I.C., and B.C.'s we find

$$f(\qquad) = f(T, T_i, T_{air}, \rho, C_p, k, t, x) = 0$$

conductivity

Since this is a differential heat transfer problem temperature is expressed as a difference and density, heat capacity, and thermal conductivity are combined to appear as $\alpha = \frac{k}{\rho c_p}$, α is known as the thermal diffusivity, which is analagous to the kinematic viscosity of fluids, and mass diffusivity.

$$\Rightarrow f(T, T_i, T_{air}, \rho, c_p, k, x, t) = f(T-T_i, T_{air}-T_i, \alpha, x, t) = 0 \quad [2]$$

Now our function consists of 5 physical quantities and 3 fundamental units (length, time, ~~mass~~, temperature). From B. π. T. there are at most 2 dimensionless variables needed for this problem. To find these variable we will use B. π. T.

$$f(T-T_{air}, T_i - T_{air}, \alpha, x, t) = 0$$

can be re-written as

$$T - T_i = g(T_{air} - T_i, \alpha, x, t)$$

$$\Rightarrow [T-T_i] = [T_{air}-T_i]^a [\alpha]^b [t]^c [x]^d$$

using the Dorsey's notation

$$\phi = [\phi]^a [L^2 T^{-1}]^b [T]^c [L]^d$$

Doing a balance on the 4 fundamental units

Temp (ϕ): $1 = a$ (a)

time (T): $0 = c - b$ (c)

length (L): $0 = 2b + d$ (d)

- We now have 3 equations and 4 unknows.

from (a), $a = 1$

From (c), $c = b$

and from (d) $d = -2b = -2c$

Choosing d as the independant power we get,

$$[T - T_i] = [T_{air} - T_i]^1 [\alpha]^{-\frac{1}{2}d} [x]^d [t]^{-\frac{1}{2}d}$$ therefore

$$\frac{[T-T_i]}{[T_{air}-T_i]} = \left[\frac{x}{\sqrt{\alpha t}}\right]^d$$

Now $$\frac{[T-T_i]}{[T_{air}-T_i]} = g\left(\frac{x}{\sqrt{\alpha t}}\right)$$

The choices for our dimensionless variables are now

$$\theta \equiv \frac{[T-T_i]}{[T_{air}-T_i]} \quad \text{and} \quad \eta \equiv \frac{x}{\sqrt{4\alpha t}}$$

Note: the factor of $\sqrt{4}$ in the denominator in η is included because this is a well studied problem and it leads to a neat solution of the original equation. Without prior knowledge of the solution we would not have used this factor.

Solving the problem:

Now our original eqn. $\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$ or $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$

By nondimensionalization becomes:

$$\frac{\partial T}{\partial t} = -\frac{1}{2}\frac{\eta}{t}\frac{d\theta}{d\eta}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{4\alpha t}\frac{d^2\theta}{d\eta^2}$$

substituting these back into our equation we get

[eq 3] $$\frac{d^2\theta}{d\eta^2} + 2\eta\frac{d\theta}{d\eta} = 0$$

Now the coorespondinng Boundry and initial conditions become

$\eta = 0, \quad \theta = 1$

$\eta = \infty, \quad \theta = 0$

One final substitution of $g = \frac{d\theta}{d\eta}$ reduces [3] into a 1st order seperable equation.

Upon integration we get: $\varphi = C_1 \exp(-\eta^2) = \frac{d\theta}{d\eta}$

a second integration gives:

$$\theta = C_1 \int_0^{\eta} \exp(-\bar{\eta}^2)\, d\bar{\eta} + C_2$$

After applying our boundry conditions $C_2 = 1$ and

$$C_1 = -\frac{1}{\int_0^{\infty} \exp(-\bar{\eta}^2)\, d\bar{\eta}} = -\frac{2}{\sqrt{\pi}}$$

$$\Rightarrow \quad \theta = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} \exp(-\bar{\eta}^2)\, d\bar{\eta} = 1 - \mathrm{erf}(\eta) = \mathrm{erfc}(\eta)$$

where erfc is the complementery error function or the gaussian distribution curve.

We can now use this relationship between temperature, depth, and time to solve our problem

We know $T_i = 20°C$, $T_{air} = -5°C$, and $T \geq 1°C$

$$\Rightarrow \quad \theta = \frac{T - T_i}{T_{air} - T_i} = \frac{-19°C}{-25°C} = 0.76$$

$\therefore$ from graph $\eta = 0.2 = \frac{x}{\sqrt{4\alpha t}}$

$t = 2\,\text{month} = 5.259488 \times 10^6\ s$

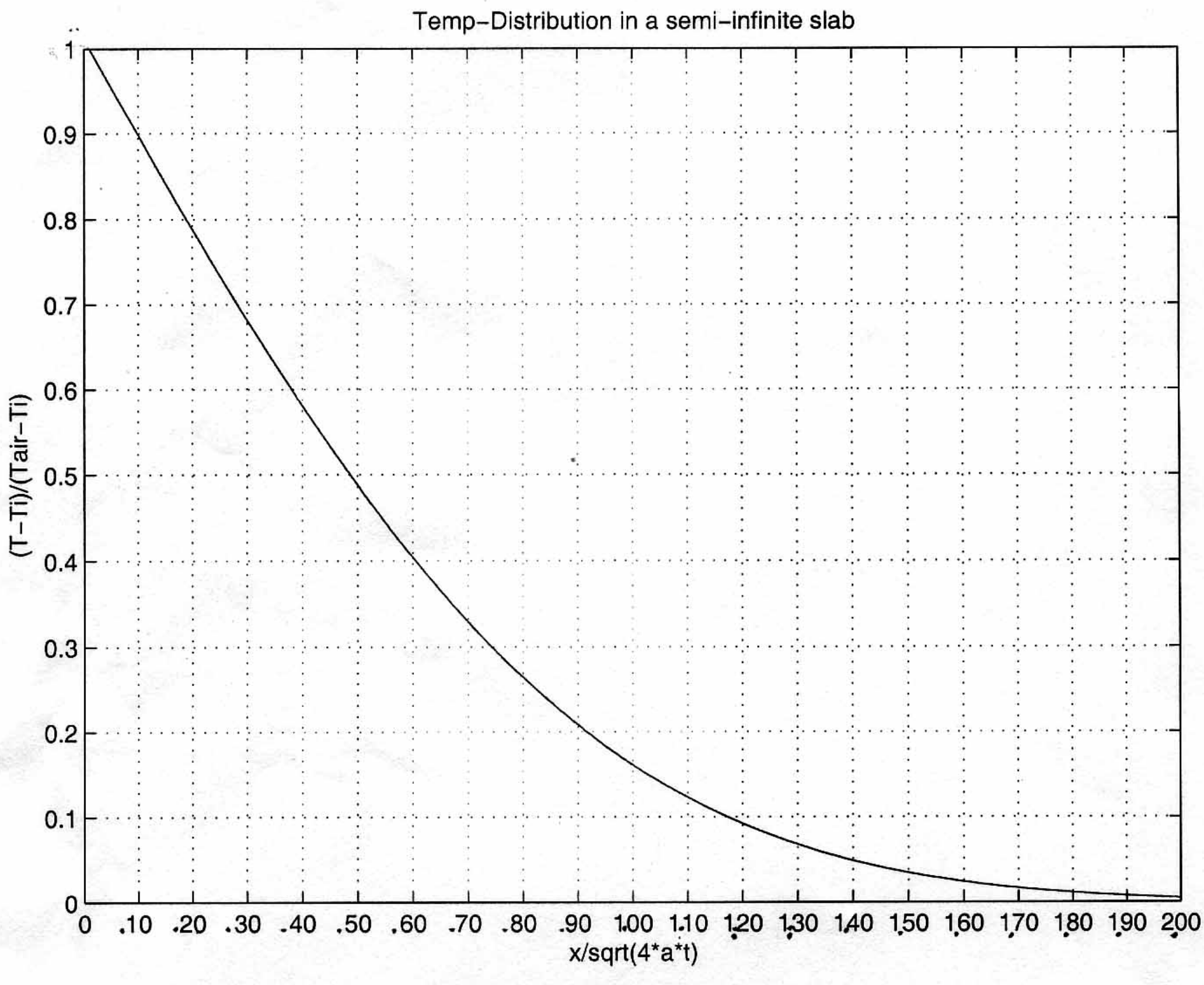
$\alpha = 0.34 \times 10^{-6}\ \frac{m^2}{s}$

$$\Rightarrow X = 0.2\sqrt{4(0.34\times10^{-6}\ \tfrac{m^2}{s})(5.259488\times10^6\ s)}$$

$$= 0.535\ m$$

$$\boxed{X \simeq 1.75\ ft}$$

This solution can analagously be applied to momentum or diffusion


Temp–Distribution in a semi–infinite slab
(T–Ti)/(Tair–Ti)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 .10 .20 .30 .40 .50 .60 .70 .80 .90 1.00 1.10 1.20 1.30 1.40 1.50 1.60 1.70 1.80 1.90 2.00
x/sqrt(4*a*t)