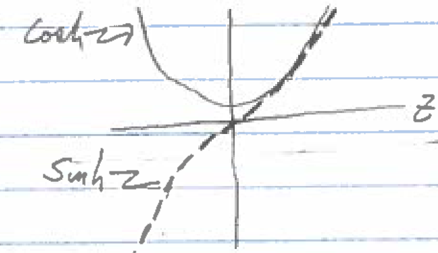
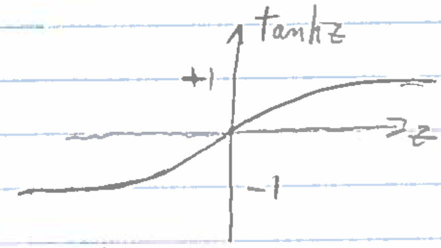
Hyperbolic Functions: $\cosh z = \frac{e^z+e^{-z}}{2}$; $\sinh z = \frac{e^z-e^{-z}}{2}$



- $\cos(iz) = \frac{e^{i(iz)}+e^{-i(iz)}}{2} = \frac{e^{z}+e^{-z}}{2} = \cosh(z)$ (1)
- $\sin(iz) = \frac{e^{i(iz)}-e^{-i(iz)}}{2i} = -\frac{e^{z}-e^{-z}}{2i} = i\sinh z$ (2)
- $\frac{d\cosh(z)}{dz} = \sinh z \qquad \frac{d\sinh(z)}{dz} = \cosh z$ (3)
- (1) & (2) $\cosh^2(z) + i^2\sinh^2(z) = \cos^2(iz) + \sin^2(iz) = 1$

  $\Rightarrow \boxed{\cosh^2(z) - \sinh^2(z) = 1}$
- $\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\cosh(z)\,dz}{\sqrt{1+\sinh^2 z}} = \int dz$ for $x = \sinh(z)$

  $= \text{arcsinh}(x)$



- define $\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$ $\Rightarrow$

  $-i\tan iz = -i\frac{\sin iz}{\cos iz} = -i\frac{i\sinh(z)}{\cosh(z)} = \tanh(z)$
- $\frac{d\tanh(z)}{dz} = \frac{d}{dz}\left(\frac{\sinh(z)}{\cosh(z)}\right) = \frac{\cosh(z)}{\cosh(z)} - \frac{\sinh(z)}{\cosh(z)^2}\sinh(z)$

  $= 1 - \frac{\sinh^2(z)}{\cosh^2(z)} = 1 - \frac{(\cosh^2(z)-1)}{\cosh^2(z)} = \frac{1}{\cosh^2 z} = \text{sech}^2(z)$
- $1-\tanh^2 z = 1/\cosh^2 z$ [compare $1+\tan^2 z = 1/\cos^2 z$]
- $\int dz\,\tanh(z) = \ln(\cosh(z))$ since $\frac{d}{dz}\ln(\cosh(z)) = \frac{1}{\cosh(z)}\sinh(z) = \tanh(z)$
- $\int \frac{dx}{1-x^2} = \int \frac{d(\tanh(z))}{1-\tanh^2 z} = \int \frac{\text{sech}^2(z)\,dz}{1-\frac{\sinh^2(z)}{\cosh^2(z)}} = \int dz$ for $x = \tanh z$

$\Rightarrow \int \frac{dx}{1-x^2} = \text{arctanh}(x)$

Recall quadratic drag $\dot{v}_y(t) = g\left(1 - \frac{v_y^2(t)}{v_{ter}^2}\right)$

for vertical motion: $m\dot{v}_y = mg - cv_y^2$, $v_{ter} = \left(\frac{mg}{c}\right)^{1/2}$

$$\Rightarrow \int_{v_{y0}}^{v_y(t)} \frac{dv_y(t')}{1 - v_y^2(t)/v_{ter}^2} = \int_0^t g\,dt' \qquad *$$

i.e. solve $\int \frac{dx}{1 - a^2x^2} = ?$

Let $ax = \tanh z$, $\Rightarrow a\,dx = \frac{1}{\cosh^2 z} dz$

$$\therefore \int \frac{dx}{1-a^2x^2} = \int \frac{1}{a}\frac{dz}{\cosh^2 z}\frac{1}{1-\tanh^2 z} = \frac{1}{a}\int \frac{dz}{\cosh^2 z\left(1 - \frac{\sinh^2 z}{\cosh^2 z}\right)}$$

$$= \frac{1}{a}\int dz$$

$$= \frac{1}{a}\tanh^{-1}(ax)$$

& $*$ $\Rightarrow$

$$\int \frac{dv_y(t')}{1 - v_y^2(t)/v_{ter}^2} = v_{ter}\tanh^{-1}\frac{v_y(t)}{v_{ter}} = gt$$

$$\Rightarrow \frac{v_y(t)}{v_{ter}} = \tanh\frac{gt}{v_{ter}}$$

$$\Rightarrow \boxed{v_y(t) = v_{ter}\tanh\left(\frac{gt}{v_{ter}}\right)}$$