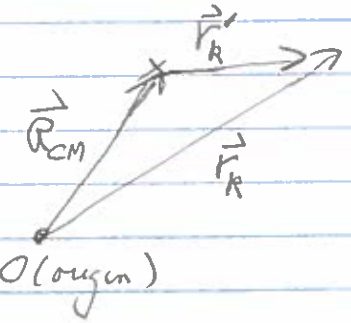
- Angular momentum of N particles w.r.t. CM ⇒ spin of system
  think spinning top.

$\vec{r}_k' = \vec{r}_k - \vec{R}_{CM}(t)$



→ r_k' defines coordinates relative to CM

→ $\vec{p}_k' = m\dot{\vec{r}}_k' \equiv$ momentum relative to CM

→ & allow $\vec{F}_{ext} \neq 0$, i.e. $\vec{F}_{ext} = M\ddot{\vec{R}}_{CM}$ (internal forces cancel)

$\vec{L}$ (with respect to origin O)

$= \sum_k' \vec{r}_k \times \vec{p}_k = \sum_k' (\vec{r}_k' + \vec{R}_{CM}(t)) \times (\vec{p}_k' + m_k \dot{\vec{R}}_{CM})$ w.r.t primed variables

$= \sum' \vec{r}_k' \times \vec{p}_k' + \sum' m_k \vec{r}_k' \times \dot{\vec{R}}_{CM} + \sum \vec{R}_{CM}(t) \times \vec{p}_k' + \sum' m_k \vec{R}_{CM} \times \dot{\vec{R}}_{CM}$

(1) (2) (3) (4)

Term (2): $\sum' m_k r_k' \times \dot{\vec{R}}_{CM} = \sum m_k (\vec{r}_k - \vec{R}_{CM}) \times \dot{\vec{R}}_{CM} = \sum' (M\vec{R}_{CM} - M\vec{R}_{CM}) \times \dot{\vec{R}}_{CM}$

$= 0$ ($\because$ $M\vec{R}_{CM} - M\vec{R}_{CM} = 0$)

Term (3): $\sum' \vec{R}_{CM} \times \vec{p}_k' = \vec{R}_{CM}(t) \times \sum' \vec{p}_k' = \vec{R}_{CM} \times (\vec{P} - M\dot{\vec{R}}_{CM}) = 0$

since $\vec{P} = M\dot{\vec{R}}_{CM}$

$\vec{L} = \sum' \vec{r}_k' \times \vec{p}_k' + M\vec{R}_{CM}(t) \times \dot{\vec{R}}_{CM}$

$= \vec{S} + \vec{R}_{CM} \times M\dot{\vec{R}}_{CM} = \vec{S} + \vec{R}_{CM} \times \vec{P}_{CM}$

= Spin + Orbit

- Calculate external torque.

$$\vec{N}_{ext} = \frac{d\vec{L}}{dt} = \frac{d\vec{S}}{dt} + \frac{d}{dt}\left(\vec{R}_{CM} \times \vec{P}_{CM}\right) = \frac{d\vec{S}}{dt} + \dot{\vec{R}}_{CM} \times \vec{P}_{CM} + \vec{R}_{CM} \times \dot{\vec{P}}_{CM}$$

$$= \frac{d\vec{S}}{dt} + \vec{R}_{CM} \times \vec{F}_{ext}$$

$$\Rightarrow \frac{d\vec{S}}{dt} = \vec{N}_{ext} - \vec{R}_{CM} \times \vec{F}_{ext} = \sum_k{}' \left(\vec{r}_k - \vec{R}\right) \times \vec{F}_{k,ext}$$

$$= \sum_k{}' \vec{r}_k' \times \vec{F}_{k,ext}$$

(time derivative of) ∑ angular momentum of a system of particles about its CM = external torque about CM

even if CM is accelerating!

see Ex. 3.4 of Taylor