(Ex) Is the Coulomb force conservative?

$$\vec{F} = k\frac{q_1 q_2}{r^2}\hat{r} \longrightarrow F' = \frac{\hat{r}}{r^2} \qquad \text{Test } \vec{\nabla}\times F' = 0 \;?$$

$k = 1/4\pi\varepsilon_0$, attractive for $|q_1| = -|q_2|$, ok

$$F' = \frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3} = \frac{1}{r^3}\left(x\hat{x} + y\hat{y} + z\hat{z}\right) \quad ; \; |r| = \sqrt{x^2+y^2+z^2}$$

$$\therefore \quad \vec{\nabla}\times F' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix}$$

$$(\nabla\times F')_x = \frac{\partial}{\partial y}\left(\frac{z}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r^3}\right) \qquad x\text{-component}$$

$$z\frac{\partial}{\partial y}\left(\frac{1}{r^3}\right) = z\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{-3/2} = -\frac{3}{2}z\left(x^2+y^2+z^2\right)^{-5/2}(2y)$$

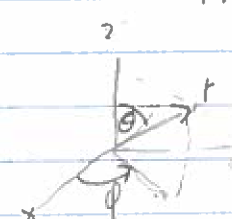
$$= -\frac{3yz}{r^5}$$

and all components are zero

$$\therefore \quad \vec{\nabla}\times\vec{F} = 0 \quad \& \quad F\,(\text{Coulomb}) \text{ is conservative}$$

$$U_2 - U_1 = -\int_{r_1}^{r_2} \vec{F}\cdot d\vec{r} = -kq_1q_2\int_{r_1}^{r_2} \frac{1}{r^2}\,\hat{r}\cdot\left(dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}\right)$$

where $d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$



$$\Rightarrow U_2 - U_1 = -kq_1q_2\int_1^2 \frac{dr}{r^2} = +kq_1q_2\,\frac{1}{r}\Big|_{r_1}^{r_2} = kq_1q_2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

Set $r_1 \to \infty$, $U_1(r_1) = 0$ $\Rightarrow$ $U(r) = \dfrac{kq_1q_2}{r}$

$k < 0$ attractive

$k > 0$ repulsive