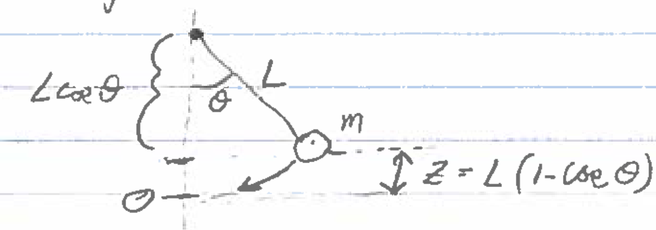
1D system not constrained to a straight line - pendulum

$v = L\dot{\theta}$
$z = L(1-\cos\theta)$



(I) $E = \frac{1}{2}mv^2 + mgz = \frac{mL^2}{2}\dot{\theta}^2 + mgL(1-\cos\theta)$ = constant!

(II) Eq of motion from $\dot{E}=0 \Rightarrow mL^2\dot{\theta}\ddot{\theta} + mgL\sin\theta\,\dot{\theta} = 0$

$\Rightarrow \dot{\theta}[mL\ddot{\theta} + mgL\sin\theta] = 0$

Two cases: $\dot{\theta} = 0$ & $[mL\ddot{\theta} + mgL\sin\theta] \neq 0 \Rightarrow$ fixed, no motion
$\dot{\theta} \neq 0$ & $[mL\ddot{\theta} + mgL\sin\theta] = 0 \Rightarrow$ oscillations

(III) Solve from (I) $\dot{\theta}^2 = \frac{2E}{mL^2} - \frac{2g}{L}(1-\cos\theta) = \frac{2}{mL^2}\left[E - mgL(1-\cos\theta)\right]$

Separation of variables } $t = \sqrt{\frac{mL^2}{2}}\int_{\theta_0}^{\theta}\frac{d\theta'}{\sqrt{E - mgL(1-\cos\theta')}}$

Simplify by taking $\theta_0 = 0$ & max angle $\theta = \theta_m$ when $\dot{\theta} = 0$

i.e. $E = mgL(1-\cos\theta_m) = 2mgL\sin^2\frac{\theta_m}{2}$ = constant

$t = \sqrt{\frac{L}{4g}}\int_0^{\theta_m}\frac{d\theta'}{\sqrt{\sin^2\theta_m/2 - \sin^2\theta'_m/2}}$

$T = 4t = 2\sqrt{\frac{L}{g}}\int_0^{\theta_m}\frac{d\theta'}{(\sin^2\theta_m/2 - \sin^2\theta'/2)^{1/2}}$ numerical solution.

Limit θ_m, θ' small

$T \approx 2\sqrt{\frac{L}{g}}\int_0^{\theta_M}\frac{d\theta'}{\left(\frac{\theta_m^2}{4} - \frac{\theta'^2}{4}\right)^{1/2}} = 4\sqrt{\frac{L}{g}}\sin^{-1}\left(\frac{\theta'}{\theta_m}\right) = 4\sqrt{\frac{L}{g}}\left(\frac{\pi}{2}\right) = 2\pi\sqrt{\frac{L}{g}}$

as expected