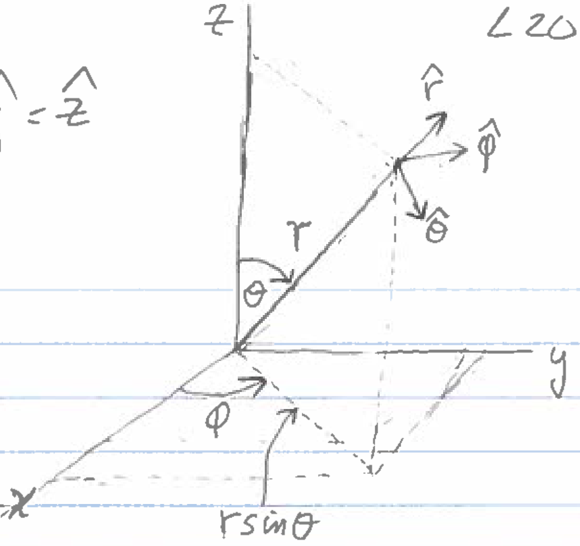
$\hat{x} \times \hat{y} = \hat{z}$



Spherical Coordinates

$\hat{r}$ points radially outwards

$\hat{\theta}$ points south along a line of longitude ($\theta \equiv$ polar ∢)

$\hat{\phi}$ points east along a line latitude ($\phi \equiv$ azimuthal angle)

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} \qquad \Rightarrow |\hat{r}| = 1$$

$$\hat{\phi} = -\cos(\pi/2 - \phi)\,\hat{x} + \sin(\pi/2 - \phi)\,\hat{y} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \qquad \Rightarrow |\hat{\phi}| = 1$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} \qquad \Rightarrow |\hat{\theta}| = 1$$

check $\hat{r} \times \hat{\theta} = \hat{\phi}$ (a Cartesian triplett, just like $\hat{i}, \hat{j}, \hat{k}$

$$(\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}) \times (\cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z})$$

skip details

$$= \cancel{\sin\theta\cos\phi\cos\theta\sin\phi\,\hat{x}\times\hat{y}} - \sin^2\theta\cos\phi\,\hat{x}\times\hat{z} + \cancel{\sin\theta\sin\phi\cos\theta\cos\phi\,\hat{y}\times\hat{x}}$$

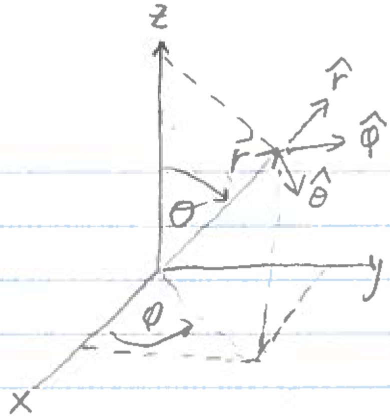
$$- \sin^2\theta\sin\phi\,\hat{y}\times\hat{z} + \cos^2\theta\cos\phi\,\hat{z}\times\hat{x} + \cos^2\theta\sin\phi\,\hat{z}\times\hat{y}$$

$$= (-\sin^2\theta\cos\phi - \cos^2\theta\cos\phi)\underbrace{\hat{x}\times\hat{z}}_{\Rightarrow -\hat{y}} - (\sin^2\theta\sin\phi + \cos^2\theta\sin\phi)\underbrace{\hat{y}\times\hat{z}}_{\Rightarrow \hat{x}}$$

$$= \cos\phi\,\hat{y} - \sin\phi\,\hat{x}$$

$$= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} = \hat{\phi} \qquad \text{checks!}$$

L20 -
-2.5



$$d\vec r = d(r\hat r) = dr\,\hat r + r\,d\hat r$$

$$\hat r = \sin\theta\cos\phi\,\hat x + \sin\theta\sin\phi\,\hat y + \cos\theta\,\hat z$$

$$\Rightarrow \underbrace{d\hat r}_{*} = (-\sin\theta\sin\phi\,d\phi + \cos\theta\cos\phi\,d\theta)\hat x$$

$d\phi$ in common $\quad$ $d\theta$ in common

$$+(\sin\theta\cos\phi\,d\phi + \cos\theta\sin\phi\,d\theta)\hat y - \sin\theta\,d\theta\,\hat z$$

$$= (-\sin\theta\sin\phi\,d\phi\,\hat x + \sin\theta\cos\phi\,d\phi\,\hat y) + (\cos\theta\cos\phi\,d\theta\,\hat x + \cos\theta\sin\phi\,d\theta\,\hat y - \sin\theta\,d\theta\,\hat z)$$

$$= \sin\theta\,d\phi(-\sin\phi\,\hat x + \cos\phi\,\hat y) + d\theta(\cos\theta\cos\phi\,\hat x + \cos\theta\sin\phi\,\hat y - \sin\theta\,\hat z)$$

$$= \sin\theta\,d\phi\,\hat\phi + d\theta\,\hat\theta$$ [compare w diagram previous pg]

$$\boxed{\begin{aligned} d\vec r &= dr\,\hat r + r\underbrace{d\hat r}_{*} = dr\,\hat r + r\,d\theta\,\hat\theta + r\sin\theta\,d\phi\,\hat\phi \\ * \quad \vec v &= \frac{d\vec r}{dt} = \dot r\,\hat r + r\dot\theta\,\hat\theta + r\sin\theta\,\dot\phi\,\hat\phi \\ * \quad v^2 &= \dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2 \end{aligned}}$$

Gradient Operator in spherical coordinates

$\vec{dr} = dr\,\hat{r} + r d\theta\,\hat{\theta} + r\sin\theta\, d\phi\,\hat{\phi}$

In Cartesian coordinates $\vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$ (I)

$df = \vec{\nabla} f \cdot \vec{dr} = \left(\hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}\right)\cdot\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right)$

$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$ Cartesian for $f = f(x,y,z)$

$= \vec{\nabla} f \cdot \vec{dr}$? for spherical coordinates $f = f(r,\theta,\phi)$

$= (\nabla f)_r\, dr + (\nabla f)_\theta\, r d\theta + (\nabla f)_\phi\, r\sin\theta\, d\phi$

$= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi$ since $f = f(r,\theta,\phi)$

$\therefore$

$(\nabla f)_r = \frac{\partial f}{\partial r}$, $(\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$, $(\nabla f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$

$\Rightarrow$ $\vec{\nabla} f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$ (II)

Check.
$\vec{\nabla} f \cdot \vec{dr} = \left(\hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\right)\cdot\left(dr\,\hat{r} + r d\theta\,\hat{\theta} + r\sin\theta\, d\phi\,\hat{\phi}\right)$

$= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi$ ✓✓✓ $f(r,\theta,\phi)$