- 2^{nd} order linear equations with constant coefficients

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x) \qquad t \to x \text{, the independent variable}$$

$$y'' + ay' + by = f(x)$$

1^{st} solve homogeneous equation (general solution <u>must</u> contain two arbitrary constants)

$$y'' + ay' + by = 0$$

(a) if $y_1(x)$ is a solution, then $c_1 y_1$ is a solution

(b) if $y_1(x)$ & $y_2(x)$ are solutions then $C_1 y_1(x) + C_2 y_2(x)$ is a solution [the general solution always has two arbitrary constants]

(c) (b) is true only if $y_1(x)$ & $y_2(x)$ are linearly independent

i.e. $\lambda y_1(x) + \mu y_2(x) = 0$ is satisfied only by $\lambda = \mu = 0$

If linearly dependent,

$$y_2(x) = \frac{-\lambda}{\mu} y_1(x) \implies \text{linear dependence}$$

which is true if $\lambda \neq 0, \mu \neq 0$

Substitute $y = e^{rx}$ into (the homogeneous eq.) $y'' + ay' + by = 0$

$\implies r^2 + ar + b = 0$ with quadratic solutions

$r_{1,2} = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4b}$ two roots satisfying $(r - r_1)(r - r_2) = 0$

Case: unequal roots $(a^2 - 4b \neq 0)$ & solution

$y = e^{r_1 x} + e^{r_2 x}$ with general solution

$\boxed{y = C_1 e^{r_1 x} + C_2 e^{r_2 x}, \quad r_1 \neq r_2}$ & $\lambda e^{r_1 x} + \mu e^{r_2 x} = 0$ for $\lambda = \mu = 0$ $\forall(x)$ in range

Case: equal roots: $a^2 = 4b$ & $r_{1,2} = \frac{-a}{2}$

$\Rightarrow \lambda e^{-a/2} + \mu e^{-a/2} = 0 \Rightarrow \lambda = -\mu$ linearly dependent

Find another solution to $y'' + ay' + by = 0$; try $y = xe^{rx}$

with $y' = e^{rx} + rxe^{rx}$, $y'' = re^{rx} + re^{rx} + r^2xe^{rx} = 2re^{rx} + r^2xe^{rx}$

$$\therefore y'' + ay' + by = 2re^{rx} + r^2xe^{rx} + ae^{rx} + arxe^{rx} + bxe^{rx}$$

$$= e^{rx}[2r + r^2x + a + arx + bx] \qquad r = -\frac{a}{2}$$

$$= e^{rx}[\cancel{-a} + \frac{a^2}{4}x \cancel{+a} - \frac{a^2}{2}x + bx]$$

$$= xe^{rx}[-\frac{a^2}{4} + b]$$

$$\equiv 0 \quad \text{since } a^2 = 4b \text{ for equal roots}$$

$$\therefore \boxed{y = c_1 e^{rx} + c_2 x e^{rx}, \quad r_1 = r_2}$$

is general solution for equal roots of auxiliary Eq. $r^2 + ar + b = 0$

✓ $\lambda e^{rx} + \mu xe^{rx} = 0$ for $\lambda = \mu = 0$

$\Rightarrow$ linear independence.