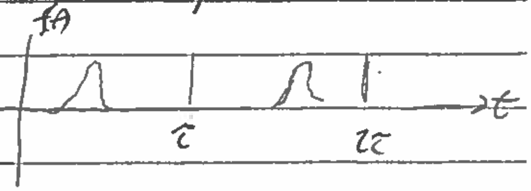
Fourier Series

Consider a function $f(t)$ that is periodic with period τ

i.e., $f(t+\tau) = f(t)$ time domain

or $g(x+\lambda) = g(x)$ spatial domain



examples: (1) musical note (pressure wave)

(2) pendulum clock (periodic impulses from detente mechanism)

(3) light house beacon (periodic intensity)

(4) EM wave (each color is different frequency)

there are many sinusoidal functions with period τ $\quad \omega = \frac{2\pi}{\tau}$

i.e.

$$\cos(n\omega t) \;\&\; \sin(n\omega t) \quad \text{for } \omega = \frac{2\pi}{\tau}, \; n = 0,1,2\cdots$$

$$\cos\left(2\pi n \frac{t}{\tau}\right) \rightarrow \cos\left[2\pi n\left(\frac{t+\tau}{\tau}\right)\right] = \cos\left[2\pi n \frac{t}{\tau} + 2\pi n\right]$$

$$= \cos\left(2\pi \frac{nt}{\tau}\right)\cos 2\pi n = \cos\left(2\pi \frac{nt}{\tau}\right) \quad n = 0,1,2\cdots$$

Jean Baptiste Fourier [1768-1830] → every τ-periodic function can be written as a linear combination of sines & cosines

i.e.,

(I) $$f(t) = \sum_{0}^{\infty}{}' \left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right] \qquad a_n, b_n \text{ real}$$

or,

(II) $$f(t) = \sum_{n=-\infty}^{\infty}{}' C_n e^{-in\omega t} \qquad C_n \text{ complex}$$

since $e^{-in\omega t} = e^{-2\pi i n \frac{t}{\tau}}$ $\quad n = 0,1,2\cdots$

and $e^{-2\pi i n\left(\frac{t+\tau}{\tau}\right)} = e^{-2\pi i n \frac{t}{\tau}} e^{-2\pi i n} = e^{-2\pi i n \frac{t}{\tau}}$ $\quad n = 0,1,2\cdots$

(I) & (II) are equivalent: Need to find

(i) relation between a_n & b_n, a_n & b_n real and C_n which is complex with real & imaginary parts.

(ii) Procedure to calculate these constants!

(i) $$\boxed{f(t) = \sum_{n=0}^{\infty}{}' \left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right] \overset{?}{=} \sum_{n=-\infty}^{\infty}{}' C_n e^{-in\omega t}}$$

$$= C_0 + \sum_{n=1}^{\infty}{}' \left[C_n e^{-in\omega t} + C_{-n} e^{in\omega t}\right] \qquad C_{-n} e^{in\omega t} = \left(C_n e^{-in\omega t}\right)^*$$

$$= C_0 + \sum_{n=1}^{\infty}{}' \left[C_n(\cos n\omega t - i \sin n\omega t) + C_{-n}(\cos n\omega t + i \sin n\omega t)\right]$$

$$= C_0 + \sum_{n=1}^{\infty}{}' \left[\underbrace{(C_n + C_{-n})}_{a_n} \cos n\omega t + \underbrace{i(-C_n + C_{-n})}_{b_n} \sin n\omega t\right]$$

& $$\boxed{\begin{aligned} a_n &= C_n + C_{-n} = C_n + C_n^* = 2\mathrm{Re}\{C_n\} \\ b_n &= -iC_n + iC_{-n} = -iC_n + iC_n^* = 2\mathrm{Im}\{C_n\} \end{aligned}}$$ since $C_{-n} = C_n^*$; since $iC_{-n} = (-iC_n)^* = iC_n^*$

Note: $-iC_n + iC_n^* = -i(x+iy) + i(x-iy) = y + y = 2y = 2\mathrm{Im}\{C_n\}$

Can solve $C_n = \frac{1}{2i}(ia_n - b_n)$; $C_{-n} = \frac{1}{2i}(ia_n + b_n)$ but not very useful

(ii) Box ⇒ $$\boxed{f(t) = \sum_{-\infty}^{\infty}{}' C_n e^{-in\omega t} \Longleftrightarrow \int_{-\tau/2}^{+\tau/2} f(t) e^{im\omega t} dt = \sum_{n=-\infty}^{\infty}{}' C_n \int_{-\tau/2}^{+\tau/2} dt\, e^{i(m-n)\omega t}}$$

evaluate

$$\int_{-\tau/2}^{\tau/2} dt\, e^{i(m-n)\omega t} = \left.\frac{e^{i(m-n)\omega t}}{i(m-n)\omega}\right|_{-\tau/2}^{+\tau/2} \qquad \text{with } \tau = \frac{2\pi}{\omega} \Rightarrow \omega\frac{\tau}{2} = \pi$$

$$= \frac{e^{i(m-n)\pi} - e^{-i(m-n)\pi}}{i(m-n)\frac{2\pi}{\tau}} = \tau\,\frac{e^{i(m-n)\pi} - e^{-i(m-n)\pi}}{2i(m-n)\pi}$$

$$= \tau\,\frac{\sin(m-n)\pi}{(m-n)\pi} \qquad \text{since } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$= \tau \frac{\sin(m-n)\pi}{(m-n)\pi} = 0 \text{ for } m \neq n$$
$$= \tau \text{ for } m = n$$

since $\lim_{x\to 0} \frac{\sin x}{x} = \frac{x + x^3/6 + \cdots}{x} = 1$

$$\Rightarrow C_m = \frac{1}{\tau}\int_{-\tau/2}^{+\tau/2} dt\, f(t)\, e^{imwt} \Longleftrightarrow f(t) = \sum_{-\infty}^{\infty}{}' C_n e^{-inwt}$$

$$\text{or } C_m = \frac{1}{\tau}\int_{0}^{\tau} dt\, f(t)\, e^{imwt} \Longleftrightarrow \quad ''$$

$$\text{with } C_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\,dt \qquad m=0 \qquad \text{Eq } 5.84$$

In terms of sines & cosines

$$a_m = 2\,\mathrm{Re}\{C_m\} = \frac{2}{\tau}\int_{-\tau/2}^{+\tau/2} f(t)\cos(m\omega t)\,dt \qquad m > 1 \;;\ \text{Eq } 5.83$$

$$b_m = 2\,\mathrm{Im}\{C_m\} = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin(m\omega t)\,dt \qquad m > 1 \;;\ \text{Eq } 5.84$$

$$\text{for } f(t) = \sum_{m=0}^{\infty}{}' \left[a_m \cos m\omega t + b_m \sin m\omega t\right] \qquad \text{Eq. } 5.82$$

Equivalently: Integrate from $0 \to \tau$ rather than $-\tau/2 \to \tau/2$

$$f(t) = \sum_{-\infty}^{\infty}{}' C_n e^{-i\omega n t} \Longleftrightarrow \int_0^{\tau} f(t)\, e^{im\omega t}\,dt = \sum_{-\infty}^{\infty}{}' C_n \int_0^{\tau} dt\, e^{i(m-n)\omega t}$$

$$\text{with } \int_0^{\tau} dt\, e^{i(m-n)\omega t} = \left.\frac{e^{i(m-n)\omega t}}{\frac{2\pi}{\tau}\, i(m-n)}\right|_0^{\tau} = \tau\,\frac{e^{i(m-n)2\pi} - 1}{2\pi i(m-n)} = 0 \text{ for } m \neq n$$
$$= 1 \text{ for } m = n$$