1-22. To evaluate $\sum_k \varepsilon_{ijk}\ \varepsilon_{\ell mk}$ we consider the following cases:

a) $i = j: \sum_k \varepsilon_{ijk}\ \varepsilon_{\ell mk} = \sum_k \varepsilon_{iik}\ \varepsilon_{\ell mk} = 0$ for all i, ℓ, m

b) $i = \ell: \sum_k \varepsilon_{ijk}\ \varepsilon_{\ell mk} = \sum_k \varepsilon_{ijk}\ \varepsilon_{imk} = 1$ for $j = m$ and $k \neq i, j$

$= 0$ for $j \neq m$

c) $i = m: \sum_k \varepsilon_{ijk}\ \varepsilon_{\ell mk} = \sum_k \varepsilon_{ijk}\ \varepsilon_{\ell ik} = 0$ for $j \neq \ell$

$= -1$ for $j = \ell$ and $k \neq i, j$

d) $j = \ell: \sum_k \varepsilon_{ijk}\ \varepsilon_{\ell mk} = \sum_k \varepsilon_{ijk}\ \varepsilon_{jmk} = 0$ for $m \neq i$

$= -1$ for $m = i$ and $k \neq i, j$

e) $j = m: \sum_k \varepsilon_{ijk}\ \varepsilon_{\ell mk} = \sum_k \varepsilon_{ijk}\ \varepsilon_{\ell jk} = 0 \text{ for } i \neq \ell$

$= 1 \text{ for } i = \ell \text{ and } k \neq i, j$

f) $\ell = m: \sum_k \varepsilon_{ijk}\ \varepsilon_{\ell mk} = \sum_k \varepsilon_{ijk}\ \varepsilon_{\ell\ell k} = 0 \text{ for all } i, j, k$

g) $i \neq \ell$ or m: This implies that $i = k$ or $i = j$ or $m = k$.

Then, $\sum_k \varepsilon_{ijk}\ \varepsilon_{\ell mk} = 0$ for all i, j, ℓ, m

h) $j \neq \ell$ or m: $\sum_k \varepsilon_{ijk}\ \varepsilon_{\ell mk} = 0$ for all i, j, ℓ, m

Now, consider $\delta_{i\ell}\,\delta_{jm} - \delta_{im}\,\delta_{j\ell}$ and examine it under the same conditions. If this behaves in the same way as the sum above, we have verified the equation

$$\sum_k \varepsilon_{ijk}\,\varepsilon_{\ell mk} = \delta_{i\ell}\,\delta_{jm} - \delta_{im}\,\delta_{j\ell}$$

a) $i = j:\ \delta_{i\ell}\,\delta_{im} - \delta_{im}\,\delta_{i\ell} = 0$ for all i, ℓ, m

b) $i = \ell:\ \delta_{ii}\,\delta_{jm} - \delta_{im}\,\delta_{ji} = 1$ if $j = m,\ i \neq j, m$

$$= 0 \text{ if } j \neq m$$

c) $i = m:\ \delta_{i\ell}\,\delta_{ji} - \delta_{ii}\,\delta_{j\ell} = -1$ if $j = \ell,\ i \neq j, \ell$

$$= 0 \text{ if } j \neq \ell$$

d) $j = \ell:\ \delta_{i\ell}\,\delta_{\ell m} - \delta_{im}\,\delta_{\ell\ell} = -1$ if $i = m,\ i \neq \ell$

$$= 0 \text{ if } i \neq m$$

e) $j = m: \ \delta_{i\ell}\,\delta_{mm} - \delta_{im}\,\delta_{m\ell} = 1 \text{ if } i = \ell,\ m \neq \ell$

$$= 0 \text{ if } i \neq \ell$$

f) $\ell = m: \ \delta_{i\ell}\,\delta_{j\ell} - \delta_{il}\,\delta_{j\ell} = 0 \text{ for all } i, j, \ell$

g) $i \neq \ell, m: \ \delta_{i\ell}\,\delta_{jm} - \delta_{im}\,\delta_{j\ell} = 0 \text{ for all } i, j, \ell, m$

h) $j \neq \ell, m: \ \delta_{i\ell}\,\delta_{jm} - \delta_{im}\,\delta_{i\ell} = 0 \text{ for all } i, j, \ell, m$

Therefore,

$$\boxed{\sum_k \varepsilon_{ijk}\,\varepsilon_{\ell mk} = \delta_{i\ell}\,\delta_{jm} - \delta_{im}\,\delta_{j\ell}}$$

Ex (4) Prove $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

$$\vec{A}\times(\vec{B}\times\vec{C})_\ell = \sum_{mn}{}' \varepsilon_{\ell mn} A_m (B\times C)_n = \sum_{mn} \varepsilon_{\ell mn} A_m \sum_{jk} \varepsilon_{njk} \underbrace{B_j C_k}$$

$$= \sum_{jkmn} \varepsilon_{\ell mn} \varepsilon_{jkn} A_m B_j C_k \qquad \text{Note different indices!}$$

$$= \sum_{jkm}{}' \left(\sum_{n}{}' \varepsilon_{\ell mn} \varepsilon_{jkn} \right) A_m B_j C_k$$

$$\Downarrow$$

$$= \sum_{jkm}{}' \left(\delta_{j\ell}\delta_{km} - \delta_{k\ell}\delta_{jm} \right) A_m B_j C_k$$

$$= \sum_{m}{}' A_m B_\ell C_m - \sum_{m}{}' A_m B_m C_\ell \qquad \text{Note collapse!}$$

$$= \left(\sum_{m}{}' A_m C_m \right) B_\ell - \left(\sum_{m}{}' A_m B_m \right) C_\ell$$

$$= (\vec{A}\cdot\vec{C}) B_\ell - (\vec{A}\cdot\vec{B}) C_\ell \qquad \ell = 1, 2, 3$$

$$\therefore \quad \vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

$$= \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \qquad [\text{BAC} - \text{CAB rule}]$$