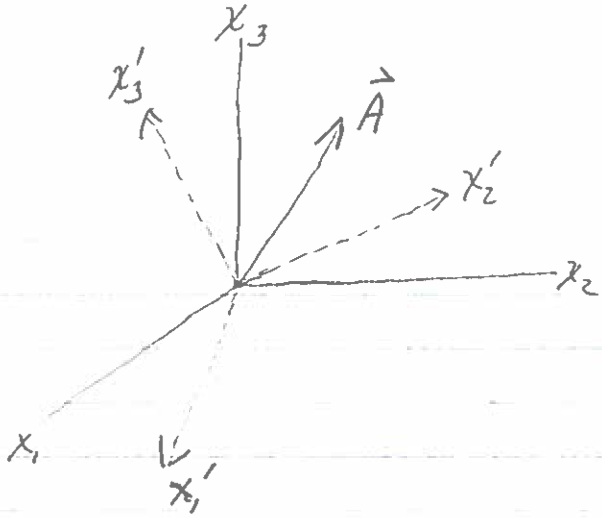

Change of coordinates – Rotation

$$\vec{A} = \sum_i{}' A_i \hat{e}_i = \sum{}' A_i' \hat{e}_i'$$

orthogonal coordinates

$$A_1' = \vec{A}\cdot\hat{e}_1' = A_1 \hat{e}_1'\cdot\hat{e}_1 + A_2 \hat{e}_1'\cdot\hat{e}_2 + A_3 \hat{e}_1'\cdot\hat{e}_3$$
$$A_2' = \vec{A}\cdot\hat{e}_2' = A_1 \hat{e}_2'\cdot\hat{e}_1 + A_2 \hat{e}_2'\cdot\hat{e}_2 + A_3 \hat{e}_2'\cdot\hat{e}_3$$
$$A_3' = \vec{A}\cdot\hat{e}_3' = A_1 \hat{e}_3'\cdot\hat{e}_1 + A_2 \hat{e}_3'\cdot\hat{e}_2 + A_3 \hat{e}_3'\cdot\hat{e}_3$$

e.g. $\hat{e}_1'\cdot\hat{e}_1$ is direction cosine of x_1' w.r.t. x_1, etc.

$$\therefore \begin{pmatrix} A_1' \\ A_2' \\ A_3' \end{pmatrix} = \begin{pmatrix} \hat{e}_1'\cdot\hat{e}_1 & \hat{e}_1'\cdot\hat{e}_2 & \hat{e}_1'\cdot\hat{e}_3 \\ \hat{e}_2'\cdot\hat{e}_1 & \hat{e}_2'\cdot\hat{e}_2 & \hat{e}_2'\cdot\hat{e}_3 \\ \hat{e}_3'\cdot\hat{e}_1 & \hat{e}_3'\cdot\hat{e}_2 & \hat{e}_3'\cdot\hat{e}_3 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \Rightarrow \vec{A}' = \tilde{R}\vec{A}$$

&

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \hat{e}_1\cdot\hat{e}_1' & \hat{e}_1\cdot\hat{e}_2' & \hat{e}_1\cdot\hat{e}_3' \\ \hat{e}_2\cdot\hat{e}_1' & \hat{e}_2\cdot\hat{e}_2' & \hat{e}_2\cdot\hat{e}_3' \\ \hat{e}_3\cdot\hat{e}_1' & \hat{e}_3\cdot\hat{e}_2' & \hat{e}_3\cdot\hat{e}_3' \end{pmatrix} \begin{matrix} A_1' \\ A_2' \\ A_3' \end{matrix} \Rightarrow \vec{A} = \tilde{R}^T \vec{A}' = \tilde{R}^{-1}\vec{A}'$$

$$\lambda_{ij} = \hat{e}_i'\cdot\hat{e}_j \Rightarrow \lambda_{ij}^T = \hat{e}_j\cdot\hat{e}_i = \lambda_{ji}$$

Recall direction cosines

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad \text{(single vector)}$$

$$\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' = \cos\theta \qquad \text{(two vectors)}$$

- For x_1'-axis in $(x_1, x_2\, x_3)$ system $(\lambda_{ij} = \hat{e}_i' \cdot \hat{e}_j)$

$$(\hat{e}_1' \cdot \hat{e}_1)^2 + (\hat{e}_1' \cdot \hat{e}_2)^2 + (\hat{e}_1' \cdot \hat{e}_3)^2 = 1$$

$\Rightarrow$

$$(\lambda_{11})^2 + (\lambda_{12})^2 + (\lambda_{13})^2 = 1$$

$\Rightarrow$

$$\sum_j{}' \lambda_{1j}^2 = \sum_j{}' \lambda_{1j}\lambda_{1j} = 1 \quad \Rightarrow \boxed{\sum_j \lambda_{ij}\lambda_{kj} = 1, \quad i = k}$$

- x_1' axis and x_2' axis are perpendicular $(\theta = \pi/2)$

$$(\hat{e}_1' \cdot \hat{e}_1)(\hat{e}_2' \cdot \hat{e}_1) + (\hat{e}_1' \cdot \hat{e}_2)(\hat{e}_2' \cdot \hat{e}_2) + (\hat{e}_1' \cdot \hat{e}_3)(\hat{e}_2' \cdot \hat{e}_3) = \cos\pi/2 = 0$$

$\Rightarrow$

$$\lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{23} = 0$$

$\Rightarrow$

$$\sum_j{}' \lambda_{1j}\lambda_{2j} = 0 \qquad \Rightarrow \boxed{\sum_j{}' \lambda_{ij}\lambda_{kj} = 0, \quad i \neq k}$$

$$\therefore \boxed{\sum_j{}' \lambda_{ij}\lambda_{kj} = \delta_{ik}} \qquad \text{orthogonality condition!}$$

Note $\sum_i{}' \lambda_{ij}\lambda_{ik} = \delta_{jk}$ is equivalent.

Matrix Multiplication - column matrix for coordinates

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \lambda_{ij} = \hat{e}_i' \cdot \hat{e}_j$$

$$\Rightarrow \quad x_i' = \sum_{ij} \lambda_{ij} x_j \quad \Rightarrow \quad \vec{x}' = \tilde{\lambda} \vec{x}$$

which is a matrix of 3 rows & 3 columns operating on a matrix of 3 rows and 1 column.

(In general $C = AB \Rightarrow C_{ij} = [AB]_{ij} = \sum_k' A_{ik} B_{kj}$

Outer (tensor) product is distinct from inner product.

$A \cdot B = \sum_i' A_i B_i$)

$$\lambda_{ij} = \hat{e}_i \cdot \hat{e}_j \Rightarrow \lambda^T_{ij} = \hat{e}_j \cdot \hat{e}_i = \lambda_{ji}$$

i.e.

$$\sum_j' \lambda_{ij} \lambda_{kj} = \sum_j' \lambda_{ij} \lambda^T_{jk} = \delta_{ik}$$

$$\Rightarrow \tilde{\lambda} \tilde{\lambda}^T = \tilde{I} \qquad I \equiv \text{identity matrix} \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

$$\Rightarrow \tilde{\lambda}^T = \tilde{\lambda}^{-1}$$

$\Rightarrow$ Transpose of orthogonal matrix is the inverse matrix!