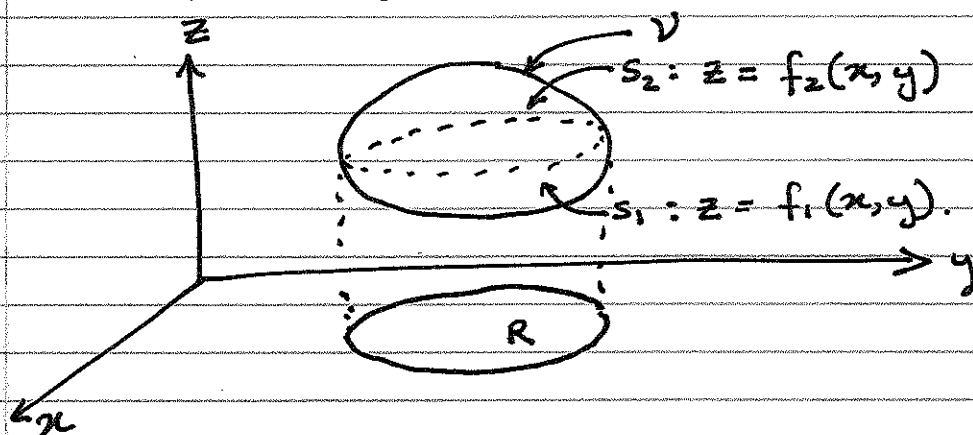


Proof of divergence theorem



The volume V is enclosed by the surface S .
The dotted line divides the volume V into two parts and S also into two parts S_1 & S_2 such that S_1 & S_2 ~~are~~ ~~single~~ can be expressed as single valued functions $S_1: z = f_1(x, y)$ & $S_2: z = f_2(x, y)$.

Also: R is the projection of S on the x, y plane

$$\text{let } \vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

$$\Rightarrow \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

To calculate $\iiint_V (\nabla \cdot \vec{A}) d\tau = \iiint_V \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) d\tau$ → this is still a triple integral

first calculate: $\int_V \frac{\partial A_z}{\partial z} d\tau = \int_V \frac{\partial A_z}{\partial z} dx dy dz$

$$= \iint_R \left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial A_z}{\partial z} dz \right] dy dx.$$

Why does this work?

Consider an integral over a sphere of radius R_0

$$\int_V d\tau = \iint_{R_0} \left[\int_{f_1}^{f_2} dz \right] dy dx.$$

$$z = f_1(x, y) = +\sqrt{R_0^2 - x^2 - y^2}$$

$$\& z = f_2(x, y) = -\sqrt{R_0^2 - x^2 - y^2}$$

$$= \iint_{R_0} (2\sqrt{R_0^2 - x^2 - y^2}) dy dx.$$

in the circular projection of the sphere on the

xy -plane (R_0): $x^2 + y^2 = R_0^2$

Switching to polar coordinates for the circular projection

$$\begin{aligned} & \int_0^{2\pi} \int_0^{R_0} 2\sqrt{R_0^2 - s^2} s ds d\varphi \quad \left(\begin{array}{l} s^2 = x^2 + y^2 \\ da = s ds d\varphi \end{array} \right) \\ &= 2 \int_0^{2\pi} d\varphi \int_0^{R_0} \sqrt{R_0^2 - s^2} s ds = 4\pi \int_0^{R_0} \sqrt{R_0^2 - s^2} s ds \end{aligned}$$

$$\text{let } R_0^2 - s^2 = u \Rightarrow -2s ds = du \Rightarrow s ds = -\frac{du}{2}$$

$$\Rightarrow \text{the integral} = -\frac{4\pi}{2} \int_{R_0^2}^0 \sqrt{u} \, du = -2\pi \left. \frac{u^{3/2}}{3/2} \right|_{R_0^2}^0$$

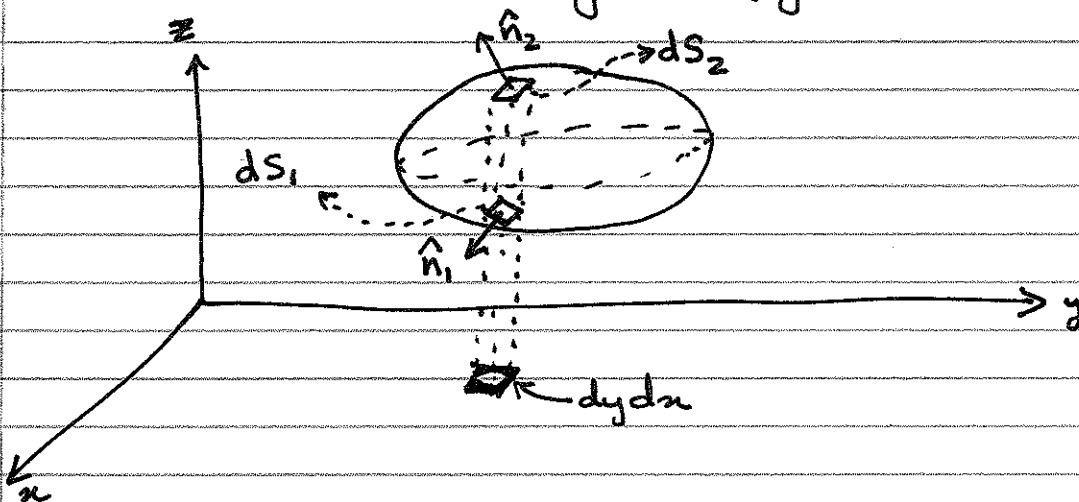
$$= -\frac{4\pi}{3} \left[0 - (R_0^2)^{3/2} \right] = \frac{4\pi}{3} R_0^3 \rightarrow \text{volume of sphere} !!$$

Going back to the original integral - - -

$$= \iint_R \left[\int_{f_1}^{f_2} \frac{\partial A_z}{\partial z} dz \right] dy dx = \iint_R \left(A_z(x, y, z) \Big|_{f_1}^{f_2} \right) dy dx$$

$$= \iint_R [A_z(x, y, f_2) - A_z(x, y, f_1)] dy dx$$

Now back to the original figure - -



$dydx$ is the projection of dS_2 on the xy -plane

$$\Rightarrow dydx = dS_2 \hat{z} \cdot \hat{n}_2 \quad \text{and} \quad dydx = -dS_1 \hat{z} \cdot \hat{n}_1$$

For dS_1 , the negative sign shows that \hat{n}_1 has a negative z -component.

$$\Rightarrow \iint_R [A_z(x, y, f_2) - A_z(x, y, f_1)] dy dx$$

$$= \iint_{S_2} A_z(x, y, z) \hat{z} \cdot \hat{n}_2 ds_2 + \iint_{S_1} A_z(x, y, z) \hat{z} \cdot \hat{n}_1 ds_1$$

$$= \iint_S A_z \hat{z} \cdot \hat{n} ds = \iint_S A_z \hat{z} \cdot d\vec{S}$$

$$\Rightarrow \iiint_V \frac{\partial A_z}{\partial z} d\tau = \oiint_S A_z \hat{z} \cdot d\vec{S} \rightarrow \textcircled{1}$$

By projecting S on the yz & zx planes we can also obtain:

$$\iiint_V \frac{\partial A_x}{\partial x} d\tau = \oiint_S A_x \hat{x} \cdot d\vec{S} \rightarrow \textcircled{2}$$

$$\text{and } \iiint_V \frac{\partial A_y}{\partial y} d\tau = \oiint_S A_y \hat{y} \cdot d\vec{S} \rightarrow \textcircled{3} \text{ respectively}$$

Adding $\textcircled{1}$, $\textcircled{2}$, & $\textcircled{3}$:

$$\iiint_V \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) d\tau = \oiint_S (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \cdot d\vec{S}$$

$$\Rightarrow \iiint_V (\nabla \cdot \vec{A}) d\tau = \oiint_S \vec{A} \cdot d\vec{S}$$