4.7. Boundary conditions

$V_{in} = V_{out}$ at $r = R$

$\cancel{\epsilon \frac{\partial V_{in}}{\partial r}}$ $\epsilon \frac{\partial V_{in}}{\partial r} = \epsilon_0 \frac{\partial V_{out}}{\partial r}$ at $r = R$

$\vec{E} = \vec{E}_0 \hat{z}$

$V_{out} \Rightarrow -E_0 z = -E_0 r\cos\theta$, for $r \gg R$

$$V_{in}(r,\theta) = \sum_l A_l r^l P_l(\cos\theta)$$

$$V_{out}(r,\theta) = -E_0 r\cos\theta + \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$\Rightarrow \sum_l A_l R^l P_l(\cos\theta) = \sum \frac{B_l}{R^{l+1}} P_l(\cos\theta) - E_0 R\cos\theta$$

$\Rightarrow$ for $l \neq 1$, $\cancel{\sum_l A_l R^l = \frac{B_l}{R^{l+1}}}$ $B_l = A_l R^{2l+1}$

for $l = 1$, $A_1 R = \frac{B_1}{R^2} - E_0 R. \Rightarrow B_1 = (A_1 + E_0) R^3$

also $\epsilon_r \sum_l l A_l R^{l-1} P_l(\cos\theta) = -E_0\cos\theta - \sum_l \frac{(l+1) B_l}{R^{l+2}} P_l(\cos\theta)$

Again: $\epsilon_r l A_l R^{l-1} = -\frac{(l+1) B_l}{R^{l+2}}$, $l \neq 1$

and $\epsilon_r A_1 = -E_0 - \frac{2B_1}{R^3}$

$\epsilon_r l \cancel{A_l} R^{l-1} = -\frac{(l+1)\cancel{A_l} R^{\cancel{2l+1} \, l-1}}{\cancel{R^{l+2}}} \Rightarrow A_l = B_l = 0$ for $l \neq 1$

$$\Rightarrow \epsilon_r A_1 = -E_0 - \frac{2(A_1+E_0)\cancel{R^3}}{\cancel{R^3}}$$

$$\Rightarrow \epsilon_r A_1 = -E_0 - 2A_1 - 2E_0 = -2A_1 - 3E_0$$

$$\Rightarrow A_1 = -\frac{3E_0}{\epsilon_r+2}, \quad B_1 = (A_1+E_0)R^3$$

$$= E_0\left(\frac{-3}{\epsilon_r+2}+1\right)R^3$$

$$= \left(\frac{\epsilon_r-1}{\epsilon_r+2}\right)E_0 R^3$$

$$\Rightarrow V_{in} = \frac{-3E_0 r\cos\theta}{\epsilon_r+2} = -\frac{3E_0 z}{\epsilon_r+2}$$

$$\vec{E} = \frac{3\vec{E}_0}{\epsilon_r+2}$$