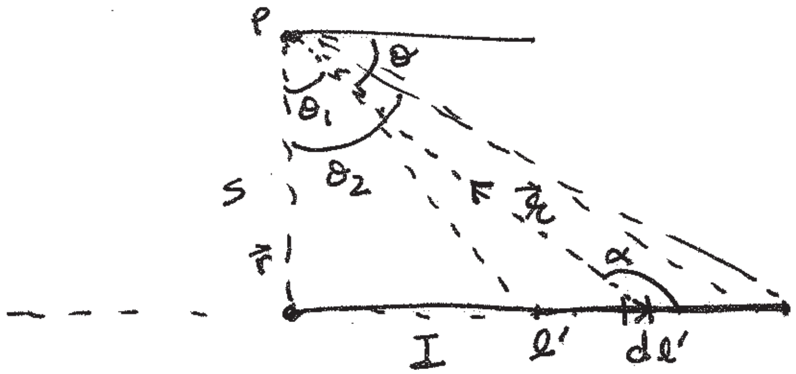
5.5.



$d\vec{l} \times \vec{\mathscr{r}}$ points out of the page

$$\frac{\pi}{2} - \theta = \pi - \alpha \Rightarrow \alpha = \frac{\pi}{2} + \theta \Rightarrow |\sin\alpha| = |\cos\theta|.$$

$$\Rightarrow \left| d\vec{l} \times \vec{\mathscr{r}} \right| = dl' \mathscr{r} \sin\alpha = dl' \mathscr{r} \cos\theta$$

$$l' = s\tan\theta \Rightarrow dl' = s\sec^2\theta \, d\theta, \quad \mathscr{r}\cos\theta = s$$

$$dB = \frac{\mu_0 I}{4\pi} \frac{dl' \cos\theta}{\mathscr{r}^2} \cancel{\Rightarrow \frac{\mu_0 I}{4\pi} s\sec^2\theta}$$

$$\Rightarrow dB = \frac{\mu_0 I}{4\pi} \frac{s\sec^2\theta.}{s^2\sec^2\theta} d\theta . \cos\theta$$

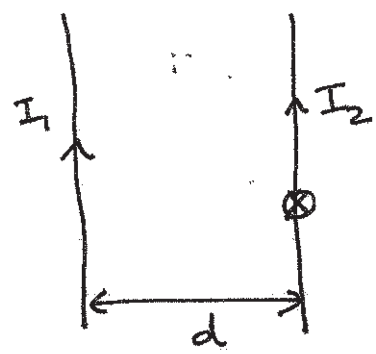
$$= \frac{\mu_0 I}{4\pi s} \cos\theta \, d\theta$$

$$\Rightarrow B = \frac{\mu_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} \cos\theta \, d\theta = \frac{\mu_0 I}{4\pi s} (\sin\theta_2 - \sin\theta_1)$$

for infinite line, $\theta_2 = \pi/2$, $\theta_1 = -\frac{\pi}{2}$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi s}.$$

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\varphi}$$

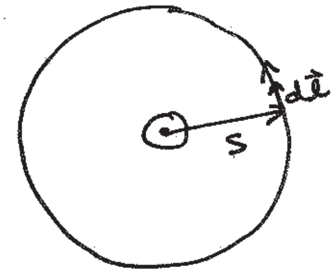


$$B = \frac{\mu_0 I_1}{2\pi d}$$

$$F = I_2 \frac{\mu_0 I_1}{2\pi d} \cdot L$$

$$\Rightarrow \frac{F}{L} = f = \frac{\mu_0 I_1 I_2}{2\pi d}$$

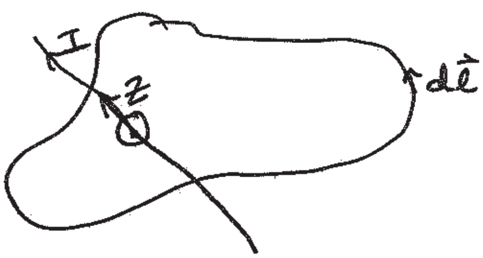
The divergence and curl of B.



$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\varphi}$$

$$\vec{B}\cdot d\vec{l} = B\,dl = \frac{\mu_0 I}{2\pi s}\,dl$$

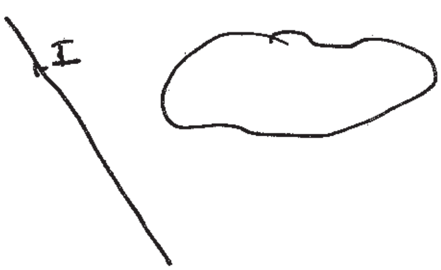
$$\oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi s}\oint dl = \mu_0 I$$



$$d\vec{l} = ds\,\hat{s} + s\,d\varphi\,\hat{\varphi} + dz\,\hat{z}$$

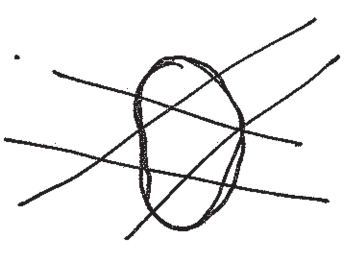
$$\vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi s}\,\cancel{ds}\,s\,d\varphi. = \frac{\mu_0 I}{2\pi}\,d\varphi$$

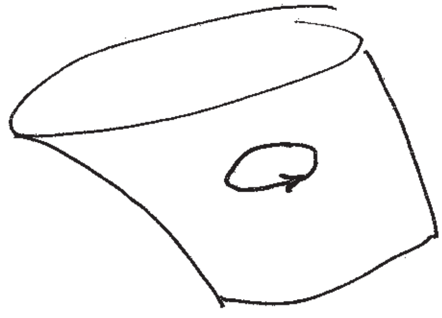
$$\Rightarrow \oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi}\oint d\varphi = \mu_0 I$$



$\rightarrow \oint d\varphi = 0$

$$\Rightarrow \oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc.}$$





$I_{enc} = \int \vec{J}\cdot d\vec{a}$

$\oint \vec{B}\cdot d\vec{l}$

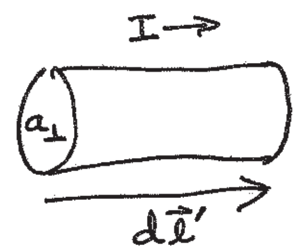
$= \int(\nabla\times\vec{B})\cdot d\vec{a}$

Stoke's

$\Rightarrow \quad \nabla \times \vec{B} = \mu_0 \vec{J} \quad \rightarrow$ not general. $\rightarrow$ general proof in make up class.

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, dl' = \frac{\mu_0}{4\pi} \int I \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$\cancel{I = J}$

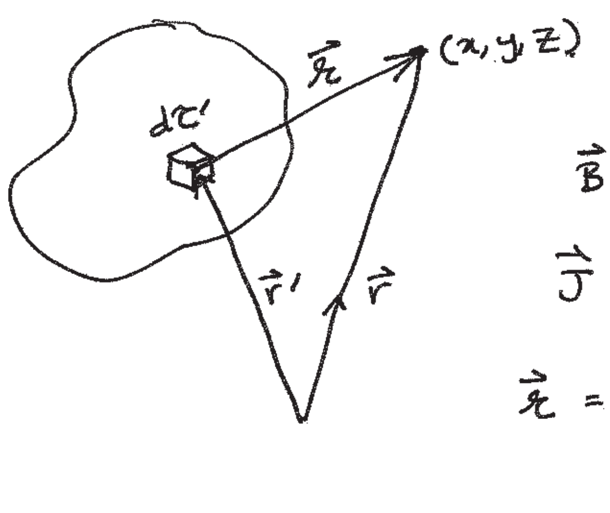


$$I\, d\vec{l}' = \vec{J} a_\perp dl'$$

$$= \cancel{\int\int \vec{J}\, da\, dl}$$

$$= \vec{J}\, d\tau'$$

$$\Rightarrow \quad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$



$\vec{B}$ is a function of $\vec{r}$ or (x, y, z)

$\vec{J}$ is a function $\vec{r}\,'$ or (x', y', z')

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$$

$$d\tau' = dx'dy'dz' \text{ or } 4\pi r'^2 \sin\theta'\, dr'd\theta'd\phi'$$

$$\nabla \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \quad \text{(unprimed coordinates)}$$

$$\Rightarrow \quad \nabla \cdot \vec{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\vec{J}(\vec{r}\,') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'$$

$$\cancel{\nabla \cdot (A_1 \times} \; \cancel{\vec{\nabla} (D} \; \vec{\nabla} \cdot (\vec{G}_1 \times \vec{G}_2) = \vec{G}_2 \cdot (\vec{\nabla} \times \vec{G}_1) - \vec{G}_1 \cdot (\nabla \times \vec{G}_2)$$

$$\Rightarrow \nabla \cdot \left[\vec{J}(\vec{r}\,') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right] = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot \underbrace{\nabla \times \vec{J}(\vec{r}\,')}_{0} - \vec{J}(\vec{r}\,') \cdot \left(\nabla \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right)$$

$$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{(x-x')\hat{x}}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}} + \;--$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ - & - & - \end{vmatrix}$$

$$\frac{\partial}{\partial y} \left\{ \frac{(z-z')}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}} \right\} = \cancel{0 \; z(z-}$$
$$= 0 - \frac{3}{\cancel{2}} [(x-x')^2 + (y-y')^2 + (z-z')^2]^{-5/2} \cdot (z-z')(y-y')$$

$$\frac{\partial}{\partial z} \left[\frac{y-y'}{- \;\; - \;\; -} \right] = -3 [(x-x')^2 + (y-y')^2 + (z-z')^2]^{-5/2} \cdot (y-y')(z-z').$$

$$\Rightarrow \quad \nabla \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 0 \quad \Rightarrow \boxed{\nabla \cdot \vec{B} = 0}$$

Remember Gauss's law: $\nabla \cdot \vec{E} = \rho/\epsilon_0$

No monopoles.

$$\left.\begin{array}{l} \nabla \cdot \vec{E} = \rho/\epsilon_0 \\ \nabla \times \vec{E} = 0 \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{B} = \mu_0 \vec{J} \end{array}\right\} \quad \vec{F} = Q(\vec{E} + \vec{v} \times \vec{B}).$$

$\nabla \cdot (\nabla \times \vec{G}) = 0$ <u>always</u>

$\Rightarrow \vec{B} = \nabla \times \vec{A}$ vector potential (Just like $\vec{E} = -\nabla V$).

But V had an ambiguity of a constant.

$\nabla V = \nabla(V + V_0)$

V_0 was set for convenience, usually so that $V \to 0$ as $r \to \infty$

$\vec{B} \neq \nabla \vec{V} \quad \because \quad \nabla \times \vec{B} \neq 0$

$\vec{B} = \nabla \times \vec{A} = \nabla \times (\vec{A} + \nabla \lambda) \quad (\nabla \times \nabla \lambda = 0)$

$\nabla \lambda$ can also be chosen for convenience.

So $\nabla \times \vec{B} = \mu_0 \vec{J}$

$$\Rightarrow \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

So we choose λ so that $\nabla \cdot \vec{A} = 0$

let $\vec{A} = \vec{A}_0 + \nabla \lambda$

$$\nabla \cdot \vec{A} = \nabla \cdot \vec{A}_0 + \nabla^2 \lambda \qquad \text{for } \nabla \cdot \vec{A} = 0$$

$$\cancel{\nabla \cdot \vec{A}_0 = \nabla^2 \lambda} \quad \nabla^2 \lambda = - \nabla \cdot \vec{A}_0 \qquad \left(\begin{array}{c} \text{similar to} \\ \nabla^2 V = -\rho/\epsilon_0 \end{array} \right)$$

$$\downarrow$$

$$V = \frac{1}{4\pi \epsilon_0} \int \frac{\rho}{\mathscr{r}} d\tau'$$

$$\Rightarrow \lambda = \frac{1}{4\pi} \int \frac{\nabla \cdot \vec{A}_0}{\mathscr{r}} d\tau'$$

if $\nabla \cdot \vec{A}_0 \rightarrow 0$ as $r \rightarrow \infty$.

$\Rightarrow$ its always possible to find a λ such that $\nabla \cdot \vec{A} = 0$

In other words $\nabla \times \vec{A} = \vec{B}$ but $\nabla \cdot \vec{A}$ can be chosen and <u>zero</u> is a convenient choice

$$\Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{J} \quad \Rightarrow \quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\vec{J}(\vec{r}')}{\mathscr{r}} d\tau'$$

if $\vec{J}(\vec{r}') \rightarrow 0$ at infinity