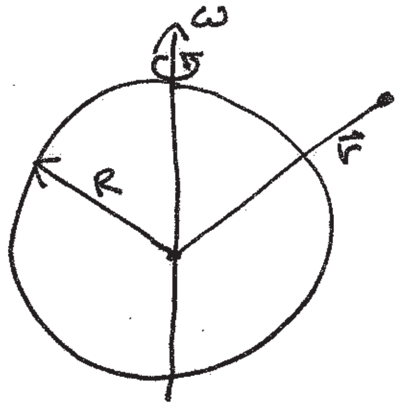
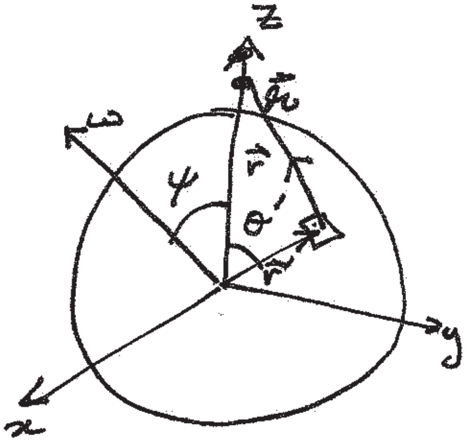
Rotating uniformly charged sphere





$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}\,')}{\mathscr{r}}\, da'$$

$\cancel{\vec{K} = \sigma\vec{v}}$, $\quad \mathscr{r} = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$

$da' = R^2 \sin\theta'\, d\theta'\, d\varphi'$

$\vec{K} = \sigma \vec{v}$

$\vec{v} = \vec{\omega} \times \vec{r}\,'$

$\vec{\omega} = \omega\sin\psi\,\hat{x} + \omega\cos\psi\,\hat{z}$

$\vec{r}\,' = R\sin\theta'\cos\varphi'\,\hat{x} + R\sin\theta'\sin\varphi'\,\hat{y} + R\cos\theta'\,\hat{z}$

$$\Rightarrow \vec{\omega}\times\vec{r}\,' = R\omega\Big[-\cos\psi\sin\theta'\underline{\sin\varphi'}\,\hat{x} + (\cos\psi\sin\theta'\underline{\cos\varphi'} - \sin\psi\cos\theta')\hat{y}$$
$$+ \sin\psi\sin\theta'\underline{\sin\varphi'}\,\hat{z}\Big]$$

· All the terms integrate to zero except: $-R\omega\sin\psi\cos\theta'\,\hat{y}$

$$\Rightarrow \vec{A}(\vec{r}) = \frac{-\mu_0 R\omega\sigma}{4\pi} \sin\psi\, R^2 \int_0^{2\pi} d\varphi' \int \frac{\cos\theta' \sin\theta'\, d\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}}\,\hat{y}$$

$$= -\frac{\mu_0 R^3 \sigma\omega \sin\psi}{2\cancel{4\pi}}\, \cancel{2\pi} \int_0^{\pi} \frac{\cos\theta' \sin\theta'\, d\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}}\,\hat{y}$$

$$\Rightarrow \vec{A}(\vec{r}) = -\frac{\mu_0 R^3 \sigma \omega \sin\psi}{2} \underbrace{\int_0^{\pi} \frac{\cos\theta' \sin\theta' \, d\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}}}_{= I} \, \hat{y}$$

$$I = \int_0^{\pi} \frac{\cos\theta' \sin\theta' \, d\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}}$$

let $u = \cos\theta' \quad \Rightarrow \quad du = -\sin\theta' d\theta'$

$$\Rightarrow I = -\int_1^{-1} \frac{u\,du}{\sqrt{R^2 + r^2 - 2Rru}} = \int_{-1}^{1} \frac{u\,du}{\sqrt{R^2 + r^2 - 2Rru}}$$

We will solve this ~~[illegible]~~ integral in the next two pages. I will ~~[illegible]~~use y & θ as ~~[illegible]~~ substitute variables and for calculating this integral y & θ have nothing to do with the y & θ coordinates of the original problem.

$$I = \int \frac{u\,du}{\sqrt{R^2+r^2-2Rru}} = \frac{1}{\sqrt{2Rr}} \int \frac{u\,du}{\sqrt{\frac{R^2+r^2}{2Rr} - u}}$$

$$u = \frac{R^2+r^2}{2Rr}\sin^2\theta = a\sin^2\theta \qquad \left(a = \frac{R^2+r^2}{2Rr}\right)$$

$$du = a.2\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow I = \frac{1}{\sqrt{2Rr}} \int \frac{2a^2\sin^3\theta\cos\theta\,d\theta}{\sqrt{a}(1-\sin^2\theta)^{1/2}} = \sqrt{\frac{2}{Rr}}\,a^{3/2}\int \sin^3\theta\,d\theta$$

$$= \sqrt{\frac{2}{Rr}}\,a^{3/2}\int (1-\cos^2\theta)\sin\theta\,d\theta, \qquad \cos\theta = y, \quad -\sin\theta\,d\theta = dy.$$

$$= -\sqrt{\frac{2}{Rr}}\,a^{3/2}\int (1-y^2)\,dy = -\sqrt{\frac{2}{Rr}}\,a^{3/2}\left[y - \frac{y^3}{3}\right]$$

$$= -\frac{a^{3/2}}{3}\sqrt{\frac{2}{Rr}}\left[3y - y^3\right] = -\frac{a^{3/2}}{3}\sqrt{\frac{2}{Rr}}\,y(3-y^2)$$

$$y = \cos\theta, \quad \sin^2\theta = \frac{u}{a} \Rightarrow y = \sqrt{1-\frac{u}{a}}$$

$$= -\frac{a^{3/2}}{3}\sqrt{\frac{2}{Rr}}\sqrt{1-\frac{u}{a}}\left(3 - 1 + \frac{u}{a}\right)$$

$$= -\frac{1}{\cancel{3}}\left(\frac{R^2+r^2}{2Rr}\right)^{3/2}\left(\frac{2}{Rr}\right)^{1/2}$$

$$= -\frac{1}{3}\,\cancel{\alpha}^{3/2}\sqrt{\frac{2}{Rr}}\;\frac{\sqrt{a-u}}{\cancel{\alpha^{1/2}}}\;\frac{(2a+u)}{\cancel{\alpha}}$$

$$= -\frac{1}{3}\sqrt{\frac{2}{Rr}}\;\sqrt{\frac{R^2+r^2}{2Rr}-u}\;\left(\frac{R^2+r^2}{Rr}+u\right)$$

$$= -\frac{1}{3}\sqrt{\frac{\cancel{2}}{Rr}}\;\frac{\sqrt{R^2+r^2-2Rru}}{\sqrt{\cancel{2}Rr}}\;\frac{(R^2+r^2+Rru)}{Rr}$$

$$= -\frac{1}{3R^2r^2}(R^2+r^2+Rru)\sqrt{R^2+r^2-2Rru}$$

$$= -\frac{(R^2+r^2+Rru)}{3R^2r^2}\sqrt{R^2+r^2-2Rru}\,\Bigg|_{-1}^{+1}$$

$$= -\frac{1}{3R^2r^2}\Big[(R^2+r^2+Rr)\,|R-r| - (R^2+r^2-Rr)(R+r)\Big]$$

if $|r|<R$

$$= -\frac{1}{3R^2r^2}\Big[\cancel{R^3}+\cancel{r^2R}+\cancel{R^2r}-\cancel{R^2r}-r^3-\cancel{Rr^2}$$
$$-\cancel{R^3}-\cancel{r^2R}-\cancel{R^2r}+\cancel{R^2r}-r^3+\cancel{Rr^2}\Big]$$

$$= +\frac{2r}{3R^2}$$

if $|r|>R$

$$= -\frac{1}{3R^2r^2}\Big[-R^3-\cancel{r^2R}-\cancel{R^2r}+\cancel{R^2r}+\cancel{r^3}+Rr^2$$
$$-R^3-\cancel{r^2R}-\cancel{R^2r}+\cancel{R^2r}-\cancel{r^3}+\cancel{Rr^2}\Big]$$

$$= \frac{2R}{3r^2}$$

$$\Rightarrow \vec{A}(\vec{r}) = -\frac{\mu_0 R^3\sigma\omega\sin\psi}{\cancel{2}}\cdot\frac{\cancel{2}r}{3R^2}\,\hat{y} \qquad (|r|<R)$$

$$\vec{\omega}\times\vec{r} = -\omega r\sin\psi\,\hat{y}$$

$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r})$$

and $\vec{A}(\vec{r}) = -\frac{\mu_0 R^3 \sigma \omega \sin\psi}{\cancel{2}} \cdot \frac{\cancel{2}R}{3r^2}\frac{r}{r}\hat{y}$

$$= \frac{\mu_0 R^4 \sigma}{3r^3}(\vec{\omega}\times\vec{r}) \quad \text{for } |\vec{r}| > R$$

Now let $\vec{\omega} = \omega\hat{z}$,

$$\Rightarrow \vec{\omega}\times\vec{r} = \omega\hat{z}\times r\hat{r} = \omega r\hat{\varphi}$$

$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0 R\sigma\omega r\sin\theta}{3}\hat{\varphi}, \quad r<R$$

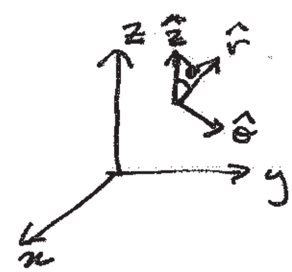
$$= \frac{\mu_0 R^4\sigma\omega\cancel{r}\sin\theta}{3r^{\cancel{3}2}}\hat{\varphi}, \quad r>R$$

$\vec{B} = \nabla\times\vec{A} \qquad (r<R)$

$$= \cancel{\frac{1}{r\sin\theta}} \quad \cancel{\frac{1}{r}} \quad \cancel{\frac{1}{r\sin\theta}\Big[}$$

$$= \frac{\mu_0 R\sigma\omega}{3}\left[\frac{1}{\cancel{r}\cancel{\sin\theta}}\cancel{r}\cdot 2\cancel{\sin\theta}\cos\theta\,\hat{r} + \frac{1}{\cancel{r}}\left(-\sin\theta\cdot 2\cancel{r}\right)\hat{\theta}\right]$$

$$= \frac{2}{3}\mu_0 R\sigma\omega\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$



$$= \frac{2}{3}\mu_0 R\sigma\omega\,\hat{z}$$